EE6604

Personal & Mobile Communications

Lecture 19

Power Spectrum of Digitally Modulated Signals

POWER SPECTRAL DENSITIES

- **A modulated signal can be written in the form**

$$\begin{aligned} s(t) &= \Re\left\{\tilde{s}(t)e^{j(2\pi f_c t+\theta_o)}\right\} \\ &= \frac{1}{2}\left\{\tilde{s}(t)e^{j(2\pi f_c t+\theta_o)} + \tilde{s}^*(t)e^{-j(2\pi f_c t+\theta_o)}\right\} \end{aligned}$$

- **The autocorrelation of the modulated signal is**

$$\begin{aligned} \phi_{ss}(\tau) &= \mathrm{E}\left[s(t+\tau)s(t)\right] \\ &= \frac{1}{4}\mathrm{E}\Big[\tilde{s}(t+\tau)\tilde{s}(t)e^{j(4\pi f_c t+2\pi f_c\tau+2\theta_o)} + \tilde{s}^*(t+\tau)\tilde{s}(t)e^{-j2\pi f_c\tau} \\ &\quad + \tilde{s}(t+\tau)\tilde{s}^*(t)e^{j2\pi f_c\tau} + \tilde{s}^*(t+\tau)\tilde{s}^*(t)e^{-j(4\pi f_c t+2\pi f_c\tau+2\theta_o)}\Big] \end{aligned}$$

- **Autocorrelation is**

$$\phi_{ss}(\tau) = \frac{1}{2}\phi^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi f_c \tau} + \frac{1}{2}\phi_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c \tau}$$

- **The power density spectrum is the Fourier transform of $\phi_{ss}(\tau)$:**

$$S_{ss}(f) = \frac{1}{2}\left[S_{\tilde{s}\tilde{s}}(f - f_c) + S^*_{\tilde{s}\tilde{s}}(-f - f_c)\right]$$

  - $S_{\tilde{s}\tilde{s}}(f)$ **is the power density spectrum of the complex low-pass signal.**

$$S_{ss}(f) = \frac{1}{2}\left[S_{\tilde{s}\tilde{s}}(f - f_c) + S_{\tilde{s}\tilde{s}}(-f - f_c)\right]$$

POWER SPECTRAL DENSITY OF A COMPLEX ENVELOPE

- **In general, the complex lowpass signal is of the form**

$$\tilde{s}(t) = A \sum_k b(t - kT, \mathbf{x}_k)$$

- **The autocorrelation of $\tilde{s}(t)$ is**

$$\begin{aligned}\phi_{\tilde{s}\tilde{s}}(t+\tau, t) &= \frac{1}{2}\mathrm{E}\left[\tilde{s}(t+\tau)\tilde{s}^*(t)\right] \\ &= \frac{A^2}{2}\sum_i \sum_k \mathrm{E}\left[b(t+\tau - iT, \mathbf{x}_i) b^*(t - kT, \mathbf{x}_k)\right] \quad .\end{aligned}$$

Observe that $\tilde{s}(t)$ is a cyclostationary random process, meaning that the autocorrelation function $\phi_{\tilde{s}\tilde{s}}(t+\tau, t)$ is periodic in t with period T. To see this property, first note that

$$\begin{aligned}&\phi_{\tilde{s}\tilde{s}}(t+T+\tau, t+T) \\ &\quad = \frac{A^2}{2}\sum_i \sum_k \mathrm{E}\left[b(t+T+\tau - iT, \mathbf{x}_i) b^*(t+T - kT, \mathbf{x}_k)\right] \\ &\quad = \frac{A^2}{2}\sum_{i'} \sum_{k'} \mathrm{E}\left[b(t+\tau - i'T, \mathbf{x}_{i'+1}) b^*(t - k'T, \mathbf{x}_{k'+1})\right] \quad .\end{aligned}$$

- **Under the assumption that the information sequence is a stationary random process we can write**

$$\begin{aligned}\phi_{\tilde{s}\tilde{s}}(t+T+\tau, t+T) &= \frac{A^2}{2}\sum_{i'}\sum_{k'} \mathrm{E}\left[b(t+\tau-i'T, \mathbf{x}_{i'})b^*(t-k'T, \mathbf{x}_{k'})\right] \\ &= \phi_{\tilde{s}\tilde{s}}(t+\tau, t) \quad . \end{aligned} \tag{1}$$

Therefore $\tilde{s}(t)$ is cyclostationary.

- **Since $\tilde{s}(t)$ is cyclostationary, the autocorrelation $\phi_{\tilde{s}\tilde{s}}(\tau)$ can be obtained by taking the time average of $\phi_{\tilde{s}\tilde{s}}(t+\tau,t)$, given by**

$$\begin{aligned}
\phi_{\tilde{s}\tilde{s}}(\tau) &= < \phi_{\tilde{s}\tilde{s}}(t+\tau,t) > \\
&= \frac{A^2}{2}\sum_i\sum_k \frac{1}{T}\int_0^T \mathrm{E}\left[b(t+\tau-iT,\mathbf{x}_i)b^*(t-kT,\mathbf{x}_k)\right]dt \\
&= \frac{A^2}{2T}\sum_i\sum_k \int_{-kT}^{-kT+T} \mathrm{E}\left[b(z+\tau-(i-k)T,\mathbf{x}_i)b^*(z,\mathbf{x}_k)\right]dz \\
&= \frac{A^2}{2T}\sum_m\sum_k \int_{-kT}^{-kT+T} \mathrm{E}\left[b(z+\tau-mT,\mathbf{x}_{m+k})b^*(z,\mathbf{x}_k)\right]dz \\
&= \frac{A^2}{2T}\sum_m\sum_k \int_{-kT}^{-kT+T} \mathrm{E}\left[b(z+\tau-mT,\mathbf{x}_m)b^*(z,\mathbf{x}_0)\right]dz \\
&= \frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty} \mathrm{E}\left[b(z+\tau-mT,\mathbf{x}_m)b^*(z,\mathbf{x}_0)\right]dz \quad .
\end{aligned}$$

where $\langle \cdot \rangle$ denotes time averaging and the second last equality used the stationary property of the data sequence $\{x_k\}$.

- **The psd of $\tilde{s}(t)$ is obtained by taking the Fourier transform of $\phi_{\tilde{s}\tilde{s}}(\tau)$,**

$$\begin{aligned}
S_{\tilde{s}\tilde{s}}(f) &= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(z+\tau-mT,\mathbf{x}_m)b^*(z,\mathbf{x}_0)dze^{-j2\pi f\tau}d\tau\right] \\
&= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty} b(z+\tau-mT,\mathbf{x}_m)e^{-j2\pi f(z+\tau-mT)}d\tau\right. \\
&\qquad \left.\times \int_{-\infty}^{\infty} b^*(z,\mathbf{x}_0)e^{j2\pi fz}dze^{-j2\pi fmT}\right] \\
&= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty} b(\tau',\mathbf{x}_m)e^{-j2\pi f\tau'}d\tau'\right. \\
&\qquad \left.\times \int_{-\infty}^{\infty} b^*(z,\mathbf{x}_0)e^{j2\pi fz}dze^{-j2\pi fmT}\right] \\
&= \frac{A^2}{2T}\sum_m \mathrm{E}\left[B(f,\mathbf{x}_m)B^*(f,\mathbf{x}_0)\right]e^{-j2\pi fmT}
\end{aligned}$$

where $B(f,\mathbf{x}_m)$ is the Fourier transform of $b(t,\mathbf{x}_m)$.

- **Finally,**

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T}\sum_m S_{b,m}(f)e^{-j2\pi fmT}$$

where

$$S_{b,m}(f) = \frac{1}{2}\mathrm{E}\left[B(f,\mathbf{x}_m)B^*(f,\mathbf{x}_0)\right]$$

- **Suppose that $\mathbf{x}_m$ and $\mathbf{x}_0$ are uncorrelated for $|m| \geq K$.**
- **Then**

$$S_{b,m}(f) = S_{b,K}(f), \quad |m| \geq K$$

where

$$\begin{aligned} S_{b,K}(f) &= \frac{1}{2}\mathrm{E}\left[B(f,\mathbf{x}_m)\right]\mathrm{E}\left[B^*(f,\mathbf{x}_0)\right] \quad |m| \geq K \\ &= \frac{1}{2}\mathrm{E}\left[B(f,\mathbf{x}_0)\right]\mathrm{E}\left[B^*(f,\mathbf{x}_0)\right] \quad |m| \geq K \\ &= \frac{1}{2}\left|\mathrm{E}\left[B(f,\mathbf{x}_0)\right]\right|^2 \ , \quad |m| \geq K \ . \end{aligned}$$

- **It follows that**

$$S_{\tilde{s}\tilde{s}}(f) = S^c_{\tilde{s}\tilde{s}}(f) + S^d_{\tilde{s}\tilde{s}}(f)$$

where

$$\begin{aligned} S^c_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T}\sum_{|m|<K}\left(S_{b,m}(f) - S_{b,K}(f)\right)e^{-j2\pi f mT} \\ S^d_{\tilde{s}\tilde{s}}(f) &= \left(\frac{A}{T}\right)^2 S_{b,K}(f)\sum_n \delta\left(f - \frac{n}{T}\right) \end{aligned}$$

ZERO MEAN SIGNALS

- **If** $\tilde{s}(t)$ **has zero mean, i.e.,** $\mathrm{E}[b(t, \underline{\mathbf{x}}_0)] = 0$**, then** $\mathrm{E}[B(f, \underline{\mathbf{x}}_0)] = 0$**.**
- **Under this condition**

$$S_{b,K}(f) = \frac{1}{2} \left| \mathrm{E}[B(f, \underline{\mathbf{x}}_0)] \right|^2 = 0$$

- **Hence,** $S_{\tilde{s}\tilde{s}}(f)$ **has no discrete component and**

$$S_{\tilde{s}\tilde{s}}(f) = \left(\frac{A^2}{T} \right) \sum_{|m|<K} S_{b,m}(f) e^{-j2\pi f m T}$$

UNCORRELATED SOURCE SYMBOLS

- **For uncorrelated source symbols**

$$b(t, \mathbf{x}_m) = b(t, x_m)$$

- **Therefore,**

$$\begin{aligned} S_{b,0}(f) &= \frac{1}{2}\mathrm{E}\left[|B(f,x_0)|^2\right] \\ S_{b,m}(f) &= \frac{1}{2}\left|\mathrm{E}\left[B(f,x_0)\right]\right|^2 \quad |m| \geq 1 \end{aligned}$$

- **Hence**

$$\begin{aligned} S^d_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T^2}S_{b,1}(f)\sum_n \delta(f - n/T) \\ S^c_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T}\left(S_{b,0}(f) - S_{b,1}(f)\right) \end{aligned}$$

- **If $\tilde{s}(t)$ has zero mean, then**

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T}S_{b,0}(f)$$

LINEAR FULL RESPONSE MODULATION

- **In this case**

$$\begin{aligned} b(t, \mathbf{x}_k) &= x_k h_a(t) \\ B(f, \mathbf{x}_k) &= x_k H_a(f) \end{aligned}$$

- **The psd of the complex envelope is**

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \left| H_a(f) \right|^2 S_{xx}(f)$$

where

$$S_{xx}(f) = \sum_m \phi_{xx}(m) e^{-j2\pi f m T}, \qquad \phi_{xx}(m) = \frac{1}{2} \mathrm{E}[x_{k+m} x_k^*]$$

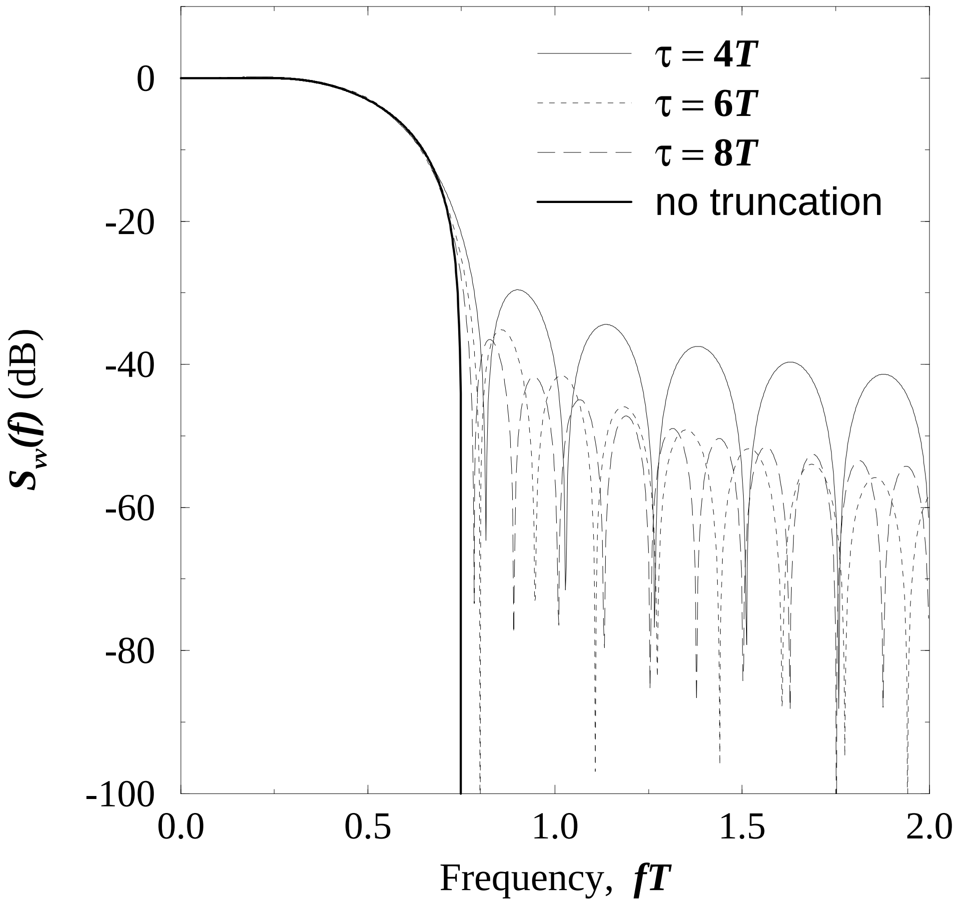
- **With uncorrelated source symbols**

$$\begin{aligned} S_{b,0}(f) &= \sigma_x^2 \left| H_a(f) \right|^2 \\ S_{b,m}(f) &= \frac{1}{2} \left| \mu_x \right|^2 \left| H_a(f) \right|^2 \quad , \quad |m| \geq 1 \quad . \end{aligned}$$

where $\mu_x = \mathrm{E}[x_m]$**. If** $\mu_x = 0$**, then** $S_{b,1}(f) = 0$ **and**

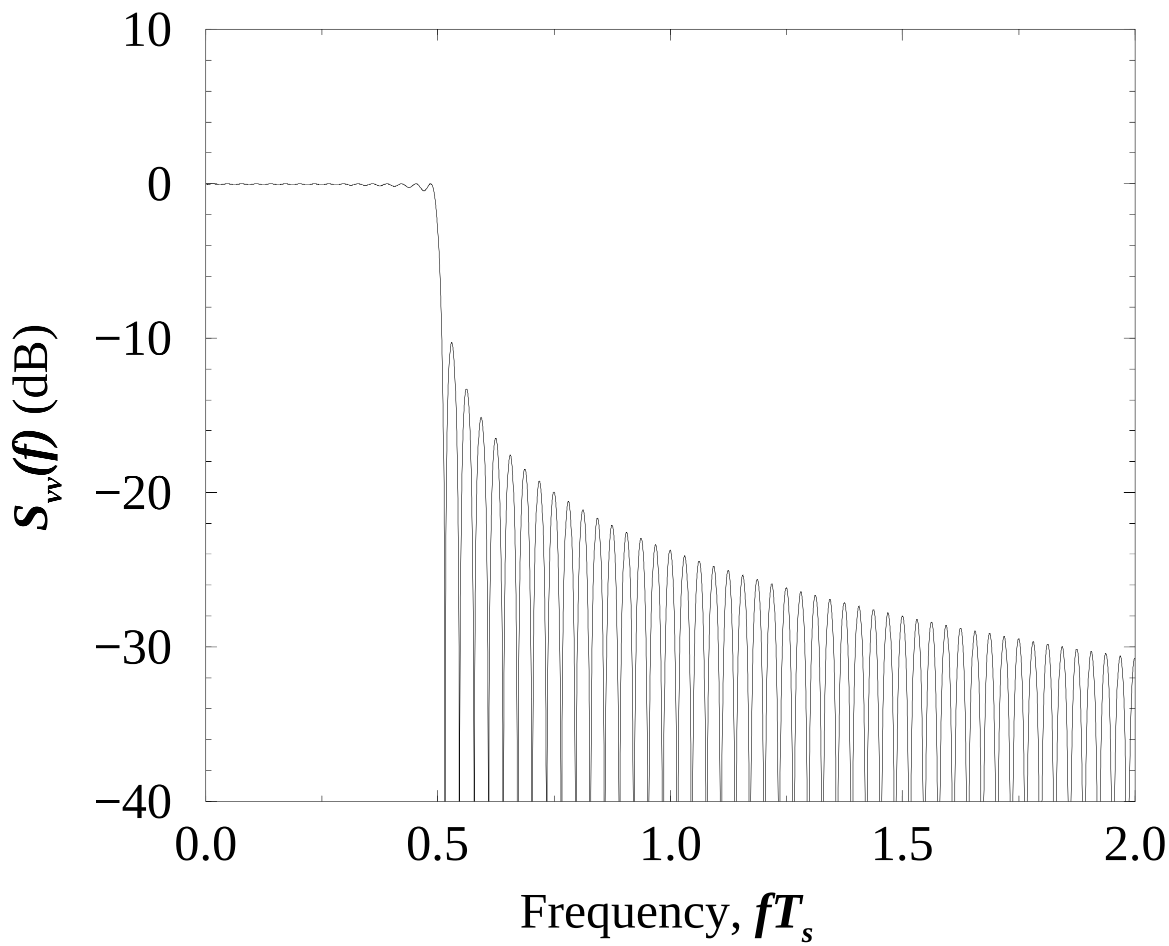
$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \sigma_x^2 \left| H_a(f) \right|^2$$

POWER SPECTRAL DENSITY OF ASK

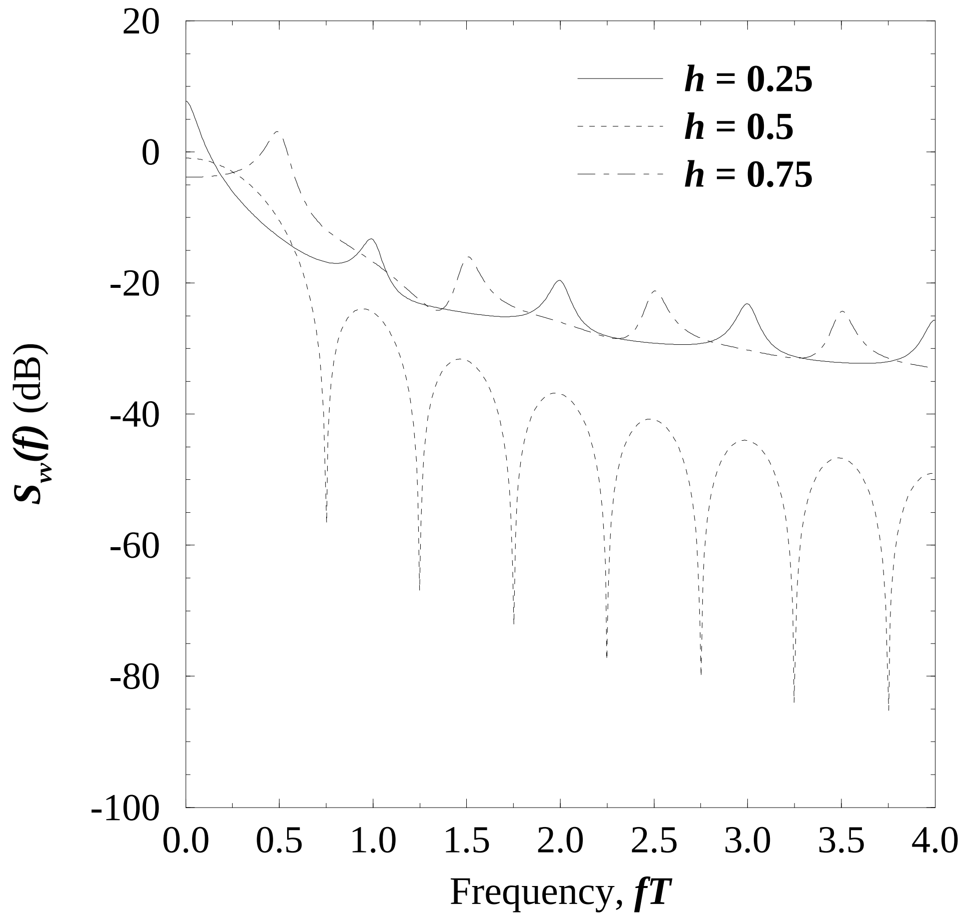


Psd of ASK with a truncated square root raised cosine pulse with various truncation lengths; $\beta = 0.5$.

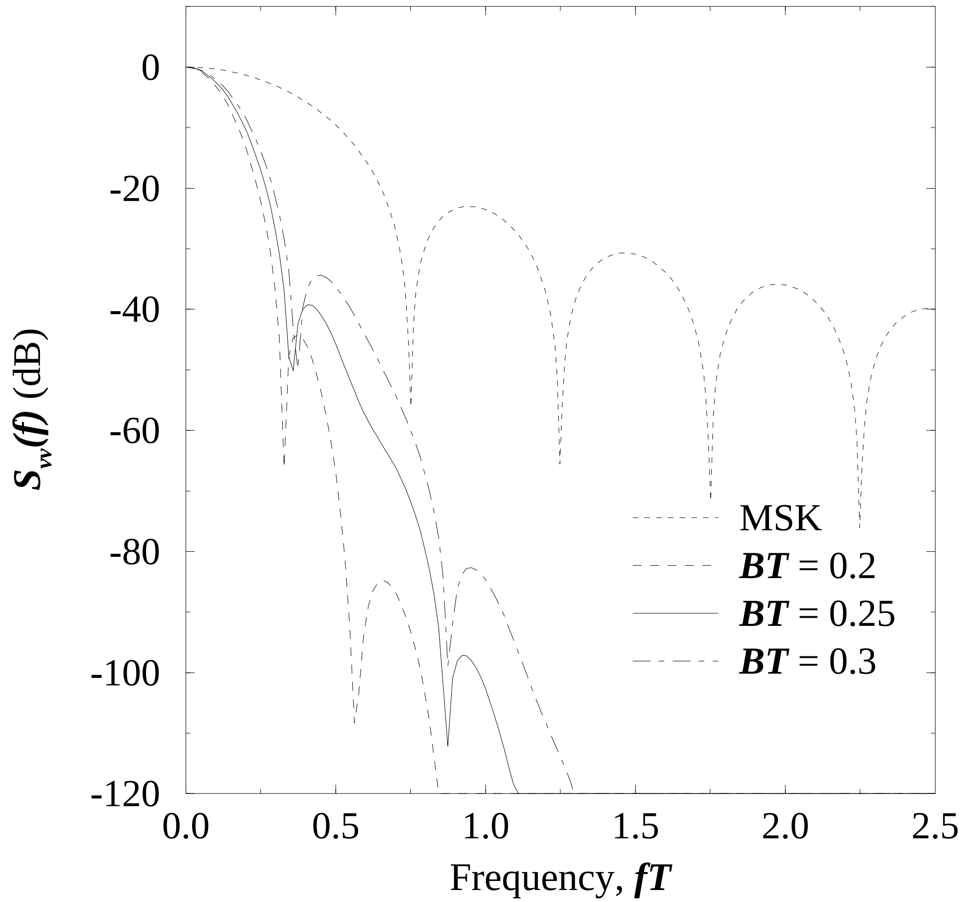
POWER SPECTRAL DENSITY OF OFDM



Psd of OFDM with $N = 32$.



Power spectral density of binary CPFSK for various modulation indices.



Power spectral density of GMSK with various normalized filter bandwidths BT.