EE6604

Personal & Mobile Communications

Lecture 23

Optimum Combining

Introduction

- **Maximum ratio combining (MRC) is the optimal combining method in a maximum likelihood sense for channels where the additive impairment is AWGN.**
- **When the additive channel impairment is dominated by co-channel interference, it is better to use optimum combining (OC) which is designed to maximize the output signal-to-interference-plus-noise ratio.**
  - **OC uses the spatial diversity not only to combat fading of the desired signal, as is the case with MRC, but also to reduce the relative power of the interfering signals at the receiver.**
  - **This is achieved by exploiting the correlation of the interference across the multiple receiver antenna elements.**
  - **By combining the signals that are received by multiple antennas, OC can suppress the interference and improve the output signal-to-interference-plus-noise ratio by several decibels.**

Received Interfering Signals

- Consider a situation of a desired signal in the presence of K co-channel interferers. The signal vectors at the L receiver antennas are equal to

$$\tilde{\mathbf{r}}_k = g_{k,0}\tilde{\mathbf{s}}_0 + \sum_{i=1}^{K} g_{k,i}\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}_k, \quad k = 1,\ \ldots,\ L \ ,$$

  where

$$\begin{aligned} \tilde{\mathbf{s}}_0 &= (\tilde{s}_{0,1}, \tilde{s}_{0,2}, \ldots, \tilde{s}_{0,N}) \\ \tilde{\mathbf{s}}_i &= (\tilde{s}_{i,1}, \tilde{s}_{i,2}, \ldots, \tilde{s}_{i,N}) \\ \tilde{\mathbf{n}}_k &= (\tilde{n}_{k,1}, \tilde{n}_{k,2}, \ldots, \tilde{n}_{k,N}) \end{aligned}$$

  are the desired signal vector, i^{th} interfering signal vector, and noise vector, respectively, N is the dimension of the signal space, and K is the number of interferers.

- The L received signal vectors can be stacked in a column to yield the $L \times N$ received matrix

$$\tilde{\mathbf{R}}_t = \mathbf{g}_0\tilde{\mathbf{s}}_0 + \sum_{i=1}^{K} \mathbf{g}_i\tilde{\mathbf{s}}_i + \tilde{\mathbf{N}} \ ,$$

  where

$$\tilde{\mathbf{R}}_t = \begin{pmatrix} \tilde{\mathbf{r}}_1 \\ \tilde{\mathbf{r}}_2 \\ \vdots \\ \tilde{\mathbf{r}}_L \end{pmatrix} , \qquad \mathbf{g}_i = \begin{pmatrix} g_{i,1} \\ g_{i,2} \\ \vdots \\ g_{i,L} \end{pmatrix} , \qquad \tilde{\mathbf{N}} = \begin{pmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{n}}_2 \\ \vdots \\ \tilde{\mathbf{n}}_L \end{pmatrix} .$$

Signal Correlations

- **The $L \times L$ received desired-signal-plus-interference-plus noise covariance matrix is given by**

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_t\tilde{\mathbf{R}}_t} = \frac{1}{2}\mathrm{E}_{\tilde{\mathbf{s}}_0,\tilde{\mathbf{s}}_i,\tilde{\mathbf{N}}}\left[\left(\mathbf{g}_0\tilde{\mathbf{s}}_0 + \sum_{i=1}^{K}\mathbf{g}_i\tilde{\mathbf{s}}_i + \tilde{\mathbf{N}}\right)\left(\mathbf{g}_0\tilde{\mathbf{s}}_0 + \sum_{i=1}^{K}\mathbf{g}_i\tilde{\mathbf{s}}_i + \tilde{\mathbf{N}}\right)^H\right] \quad , \tag{1}$$

  where $(\ \cdot\ )^H$ denotes complex conjugate transpose.

- **Likewise, the received interference-plus-noise covariance matrix is given by**

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} = \frac{1}{2}\mathrm{E}_{\tilde{\mathbf{s}}_i,\tilde{\mathbf{N}}}\left[\left(\sum_{i=1}^{K}\mathbf{g}_i\tilde{\mathbf{s}}_i + \tilde{\mathbf{N}}\right)\left(\sum_{i=1}^{K}\mathbf{g}_i\tilde{\mathbf{s}}_i + \tilde{\mathbf{N}}\right)^H\right] \quad .$$

- **Note that the expectations are taken over a period that is much less than the channel coherence time, i.e., several modulated symbol durations.**

- **If the interfering signal and noise vectors are uncorrelated**

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_t\tilde{\mathbf{R}}_t} = \mathbf{g}_0\mathbf{g}_0^H E_{\mathrm{av}} + \sum_{i=1}^{K}\mathbf{g}_i\mathbf{g}_i^H E_{\mathrm{av}}^i + N_o\mathbf{I} \quad , \tag{2}$$

  and

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} = \sum_{i=1}^{K}\mathbf{g}_i\mathbf{g}_i^H E_{\mathrm{av}}^i + N_o\mathbf{I} \quad ,$$

  respectively, where $\mathbf{I}$ is the $L \times L$ identity matrix and E_{av}^i is the average energy in the i^{th} interfering signal.

- **The matrices $\mathbf{\Phi}_{\tilde{\mathbf{R}}_t\tilde{\mathbf{R}}_t}$ and $\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}$ will vary at the channel fading rate.**

Optimum Combining and MMSE Solution

- **The received signals vectors $\tilde{\mathbf{r}}_k$ are multiplied by controllable weights w_k and summed together, i.e., the combiner output is**

$$\tilde{\mathbf{r}} = \sum_{k=1}^{L} w_k \tilde{\mathbf{r}}_k = \mathbf{w}^T \tilde{\mathbf{R}}_t \quad ,$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_L)^T$ is the weight vector.

- **Several approaches can be taken to find the optimal weight vector w. One approach is to minimize the mean square error**

$$\begin{aligned} J &= \mathrm{E}\left[\|\tilde{\mathbf{r}} - \tilde{\mathbf{s}}_0\|^2\right] \\ &= \mathrm{E}\left[\|\mathbf{w}^T \tilde{\mathbf{R}}_t - \tilde{\mathbf{s}}_0\|^2\right] \\ &= 2\mathbf{w}^T \mathbf{\Phi}_{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}_t} \mathbf{w}^* - 4\mathrm{Re}\left\{\mathbf{\Phi}_{\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t} \mathbf{w}^*\right\} - 2E_{\mathrm{av}} \end{aligned}$$

where $\mathbf{\Phi}_{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}_t}$ is defined in (1) and

$$\mathbf{\Phi}_{\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t} = \mathrm{E}\left[\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t^H\right] = 2E_{\mathrm{av}} \mathbf{g}_0^* \quad .$$

- **The minimum mean square error (MMSE) solution is obtained by**

$$\nabla_{\mathbf{w}} J = \left(\frac{\partial J}{\partial w_1}, \ \cdots, \ \frac{\partial J}{\partial w_L}\right) = 4\mathbf{w}^T \mathbf{\Phi}_{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}_t} - 4\mathbf{\Phi}_{\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t} = 0 \quad .$$

The solution is

$$\mathbf{w}_{\mathrm{opt}} = \mathbf{\Phi}_{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}_t}^{-1} \mathbf{\Phi}_{\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t}^T = 2E_{\mathrm{av}} \mathbf{\Phi}_{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}_t}^{-1} \mathbf{g}_0^*$$

where we have used the fact that $\mathbf{\Phi}_{\tilde{\mathbf{s}}_0 \tilde{\mathbf{R}}_t}^T = 2\mathbf{g}_0^* E_{\mathrm{av}}$.

Optimum Combining and MMSE Solution

- **Since $\boldsymbol{\Phi}_{\tilde{\mathbf{R}}_t\tilde{\mathbf{R}}_t} = \mathbf{g}_0\mathbf{g}_0^H E_{\text{av}} + \boldsymbol{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}$, we can write**

$$\begin{aligned}\mathbf{w}_{\text{opt}} &= 2E_{\text{av}}\left(\boldsymbol{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} + \mathbf{g}_0\mathbf{g}_0^H E_{\text{av}}\right)^{-1}\mathbf{g}_0^* \\ &= 2E_{\text{av}}\left(\boldsymbol{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} + \mathbf{g}_0^*\mathbf{g}_0^T E_{\text{av}}\right)^{-1}\mathbf{g}_0^* \quad .\end{aligned}$$

- **Next, we apply a variation of the matrix inversion lemma**

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{1 + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}}$$

resulting in

$$\begin{aligned}\mathbf{w}_{\text{opt}} &= 2E_{\text{av}}\left(\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} - \frac{E_{\text{av}}\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^*\mathbf{g}_0^T\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}}{1 + E_{\text{av}}\mathbf{g}_0^T\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^*}\right)\mathbf{g}_0^* \\ &= 2E_{\text{av}}\left(\frac{1}{1 + E_{\text{av}}\mathbf{g}_0^T\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^*}\right)\cdot\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^* \\ &= C\cdot\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^* \quad ,\end{aligned}$$

where $C = 2E_{\text{av}}/(1 + E_{\text{av}}\mathbf{g}_0^T\boldsymbol{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0^*)$ is a scalar.

Optimum Combining and Maximum SINR Solution

- Another criterion is to maximize the instantaneous signal-to-interference-plus-noise ratio (SINR) at the output of the combiner

$$\omega = \frac{\mathbf{w}^T \mathbf{g}_0 \mathbf{g}_0^H E_{\mathrm{av}} \mathbf{w}^*}{\mathbf{w}^T \mathbf{\Phi}^{-1}_{\tilde{\mathbf{R}}_i \tilde{\mathbf{R}}_i} \mathbf{w}^*} \quad .$$

- Solving for the optimum weight vector gives

$$\mathbf{w}_{\mathrm{opt}} = B \cdot \mathbf{\Phi}^{-1}_{\tilde{\mathbf{R}}_i \tilde{\mathbf{R}}_i} \mathbf{g}_0^* \quad ,$$

  where B is an arbitrary constant. Hence, the *maximum* instantaneous output SINR is

$$\omega = E_{\mathrm{av}} \mathbf{g}_0^H \mathbf{\Phi}^{-1}_{\tilde{\mathbf{R}}_i \tilde{\mathbf{R}}_i} \mathbf{g}_0 \quad .$$

- Note that the maximum instantaneous output SINR does not depend on the choice of the scalar B. Therefore, the MMSE weight vector also maximizes the instantaneous output SINR.

- When no interference is present, $\mathbf{\Phi}_{\tilde{\mathbf{R}}_i \tilde{\mathbf{R}}_i} = N_o \mathbf{I}$ and

$$\mathbf{w}_{\mathrm{opt}} = \frac{\mathbf{g}_0^*}{N_o} \quad ,$$

  so that the combiner output is

$$\tilde{\mathbf{r}} = \sum_{k=1}^{L} \frac{g_{0,k}^*}{N_o} \tilde{\mathbf{r}}_k \quad .$$

  – OC reduces to MRC when no interference is present.

Performance of Optimum Combining

- **To evaluate the performance of OC, several definitions are required as follows:**

$$\begin{aligned}
\Omega &= \frac{\textbf{average received desired signal power per antenna}}{\textbf{average received noise plus interference power per antenna}} \\
\bar{\gamma}_c &= \frac{\textbf{average received desired signal power per antenna}}{\textbf{average received noise power per antenna}} = \frac{\mathrm{E}[|g_{0,k}|^2]E_{\mathrm{av}}}{N_o} \\
\bar{\gamma}_i &= \frac{\textbf{average received } i^{\mathrm{th}} \textbf{ interferer power per antenna}}{\textbf{average received noise power per antenna}} = \frac{\mathrm{E}[|g_{i,k}|^2]E^i_{\mathrm{av}}}{N_o} \\
\omega_R &= \frac{\textbf{instantaneous desired signal power at the array output}}{\textbf{average noise plus interference power at the array output}} \\
\omega &= \frac{\textbf{instantaneous desired signal power at the array output}}{\textbf{instantaneous noise plus interference power at the array output}}
\end{aligned}$$

- **In the above definitions, "average" refers to the average over the Rayleigh fading, while "instantaneous" refers to an average over a period that is much less than the channel coherence time, i.e., several modulated symbol durations.**

- **Note that**

$$\Omega = \frac{\bar{\gamma}_c}{1 + \Sigma_{k=1}^{K} \bar{\gamma}_i} \quad .$$

Fading of the Desired Signal Only

- ω_R **is equal to**

$$\omega_R = E_{\rm av}\mathbf{g}_0^H \mathbf{\Phi}^{-1}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}\mathbf{g}_0 \quad ,$$

where, with a single interferer,

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} = E^1_{\rm av}{\rm E}[\mathbf{g}_1\mathbf{g}_1^H] + N_o\mathbf{I} \quad .$$

Note that the above expectation in is over the Rayleigh fading. The pdf of ω_R **is**

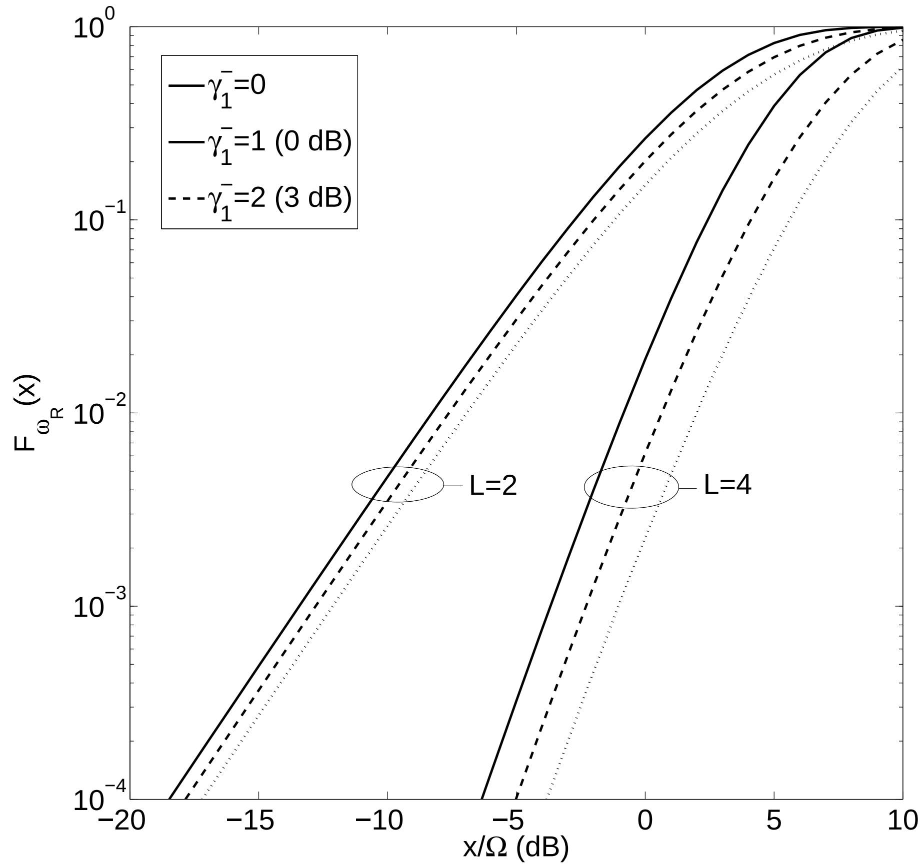
$$p_{\omega_R}(x) = \frac{e^{-x/\bar{\gamma}_c}(x/\bar{\gamma}_c)^{L-1}(1+L\bar{\gamma}_1)}{\bar{\gamma}_c(L-2)!}\int_0^1 e^{-((x/\bar{\gamma}_c)L\bar{\gamma}_1)t}(1-t)^{L-2}dt \tag{3}$$

and the cdf of ω_R **is**

$$F_{\omega_R}(x) = \int_0^{x/\bar{\gamma}_c} \frac{e^{-y}y^{L-1}(1+L\bar{\gamma}_1)}{(L-2)!}\int_0^1 e^{-(yL\bar{\gamma}_1)t}(1-t)^{L-2}dtdy \quad . \tag{4}$$

which are valid for $L \geq 2$**.**

- **Note that** ω_R **in (3) and (4) is normalized by** $\bar{\gamma}_c$**. Since** $\bar{\gamma}_c = (1+\bar{\gamma}_1)\Omega$ **for the case of a single interferer, it is apparent that** ω_R **can be normalized by** Ω **as well. The normalization by** Ω **allows for a straight forward comparison of OC and MRC.**



Cdf of γ_s^{mr} *for maximal ratio combining;*
$\bar{\gamma}_c$ *is the average branch symbol energy-to-noise ratio.*

BER Performance

- **The probability of bit error for coherently detected BPSK is given by**

$$P_b = \int_0^\infty Q\left(\sqrt{2x}\right) p_{\omega_R}(x)dx$$

- **Bogachev and Kieslev derived the bit error probability (for $L \geq 2$) as**

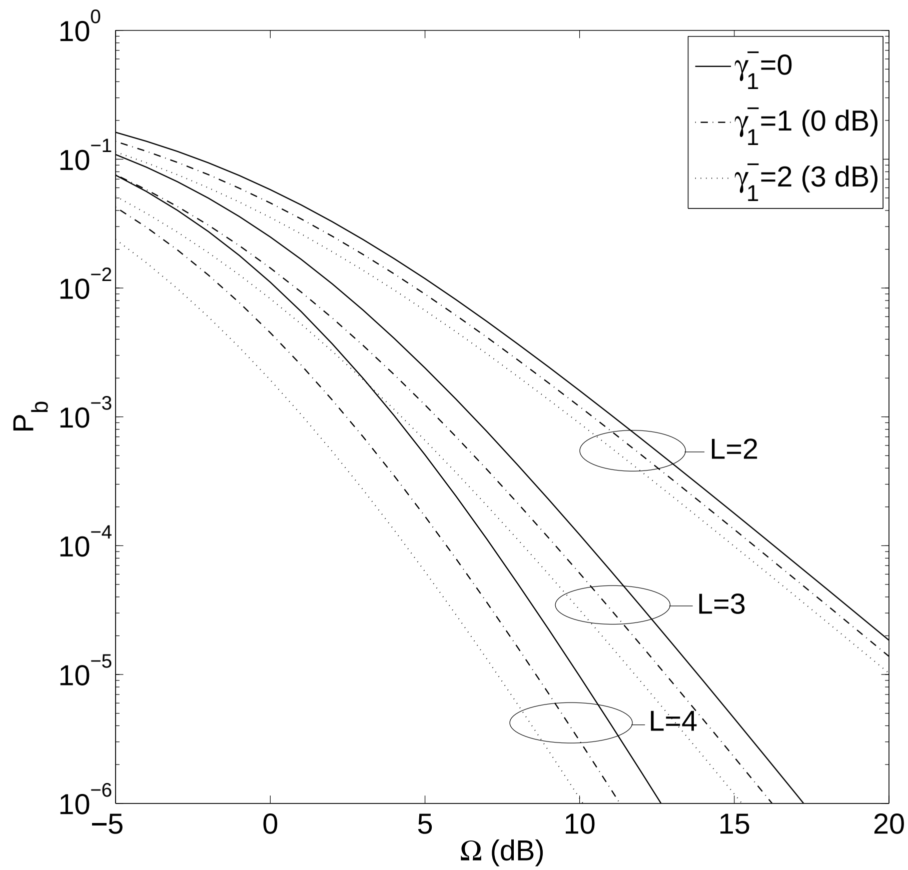
$$\begin{aligned} P_b \;=\; & \frac{(-1)^{L-1}(1+L\bar{\gamma}_1)}{2(L\bar{\gamma}_1)^{L-1}} \left\{ -\frac{L\bar{\gamma}_1}{1+L\bar{\gamma}_1} + \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} - \frac{1}{1+L\bar{\gamma}_1}\sqrt{\frac{\bar{\gamma}_c}{1+L\bar{\gamma}_1+\bar{\gamma}_c}} \right. \\ & \left. - \sum_{k=1}^{L-2}(-L\bar{\gamma}_1)^k \left[1 - \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} \left(1 + \sum_{i=1}^{k} \frac{(2i-1)!!}{i!(2+2\bar{\gamma}_c)^i} \right) \right] \right\} \end{aligned}$$

where

$$(2i-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2i-1) \quad .$$

- **Simon and Alouini have derived the following expression which is valid for $L \geq 1$:**

$$\begin{aligned} P_b \;=\; & \frac{1}{2}\left\{ 1 - \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} \sum_{k=0}^{L-2} \binom{2k}{k} \frac{1}{[4(1+\bar{\gamma}_c)]^k} \left[1 - \left(-\frac{1}{L\bar{\gamma}_1} \right)^{L-1-k} \right] \right. \\ & \left. - \sqrt{\frac{\bar{\gamma}_c}{1+L\bar{\gamma}_1+\bar{\gamma}_c}} \left(-\frac{1}{L\bar{\gamma}_1} \right)^{L-1} \right\} \end{aligned}$$



Bit error probability for coherent BPSK and optimal combining for various values of $\bar{\gamma}_1$ *and various number of receiver antenna elements, L.*

Fading of the Desired and Interfering Signals

- **The maximum instantaneous output SINR is equal to**

$$\omega = E_{\rm av}\mathbf{g}_0^H \mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}^{-1}\mathbf{g}_0 \ \ ,$$

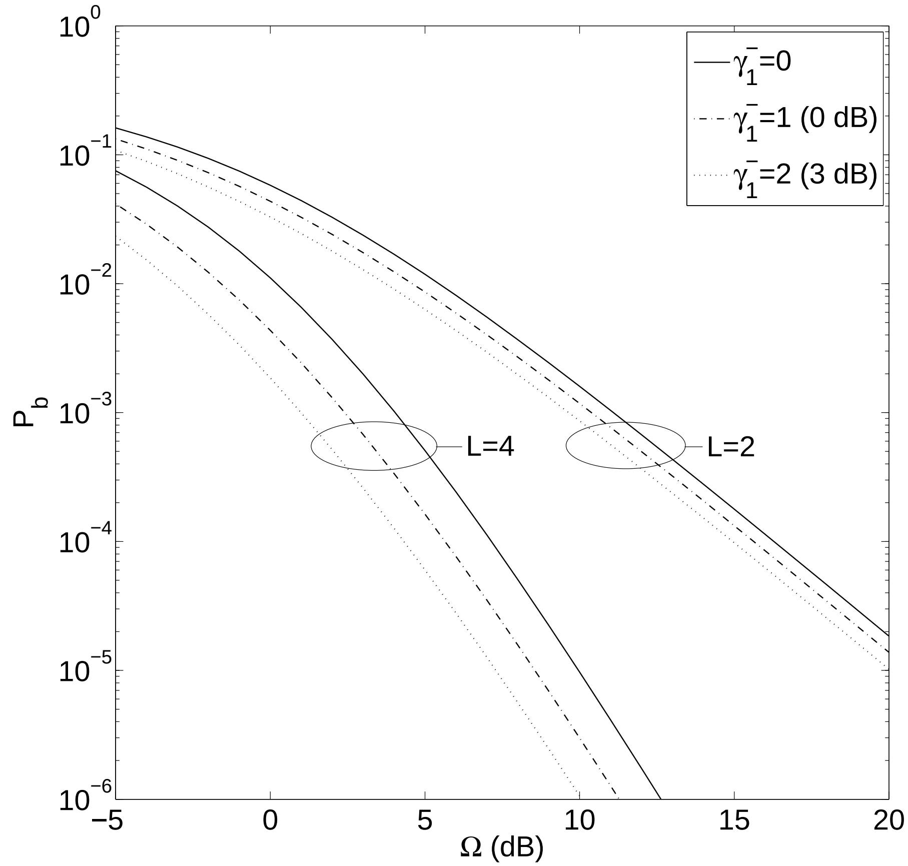
where, with a single interferer,

$$\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i} = E_{\rm av}^1 \mathbf{g}_1\mathbf{g}_1^H + N_o\mathbf{I} \ \ .$$

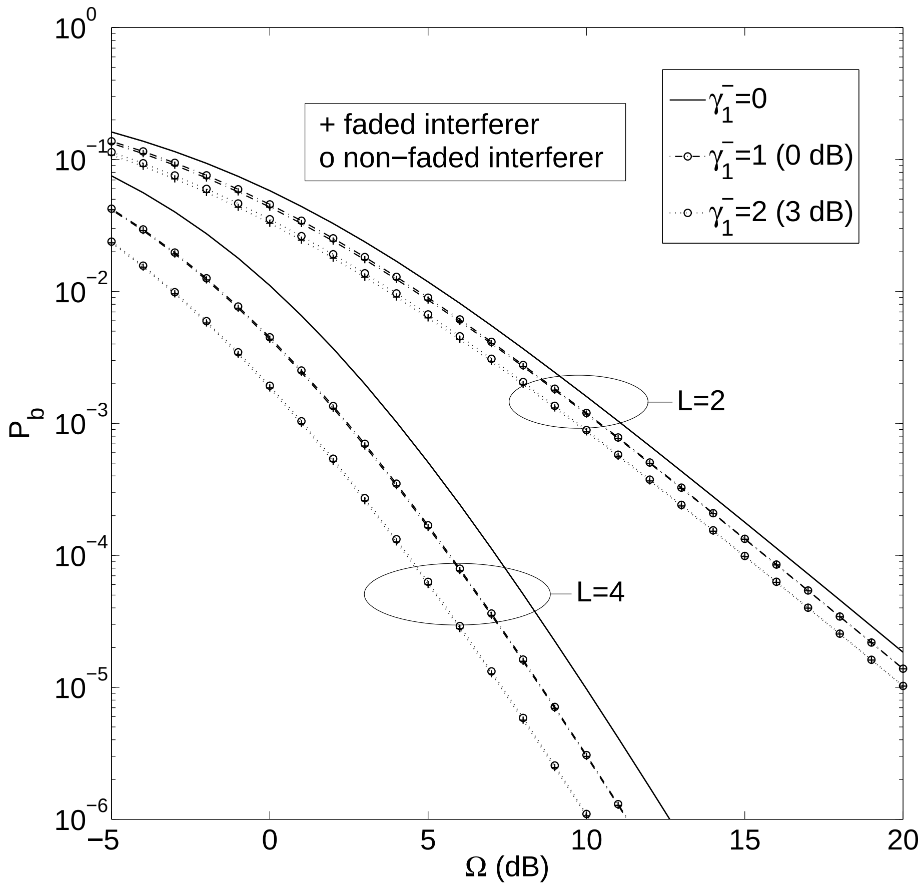
In this case, the matrix $\mathbf{\Phi}_{\tilde{\mathbf{R}}_i\tilde{\mathbf{R}}_i}$ varies at the fading rate.

- **Using eigenvalue decomposition, the probability of bit error is**

$$\begin{aligned} P_b &= \int_0^\infty P_{b|\gamma_1}(x) p_{\gamma_1}(x)dx \\ &= \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_c}{\bar{\gamma}_c+1}}\sum_{k=0}^{L-2}\binom{2k}{k}\left(\frac{1}{4(\bar{\gamma}_c+1)}\right)^k\right] \\ &\quad - \frac{1}{2\Gamma(L)(-\bar{\gamma}_1)^{L-1}}\left\{\sqrt{\frac{\pi\bar{\gamma}_c}{\bar{\gamma}_1}}\exp\left(\frac{\bar{\gamma}_c+1}{\bar{\gamma}_1}\right)\mathrm{erfc}\left(\sqrt{\frac{\bar{\gamma}_c+1}{\bar{\gamma}_1}}\right)\right. \\ &\quad \left. - \sqrt{\frac{\bar{\gamma}_c}{\bar{\gamma}_c+1}}\sum_{k=0}^{L-2}\frac{(2k)!}{k!}\left(\frac{-\bar{\gamma}_1}{4(\bar{\gamma}_c+1)}\right)^k\right\} \ \ . \end{aligned}$$



Bit error probability for coherent BPSK and optimal combining for various values of $\bar{\gamma}_1$ and various number of receiver antenna elements, L.



Comparison of the bit error probability for coherent BPSK and optimal combining for a non-faded interferer and a faded interferer; the performance is almost identical.