EE6604

Personal & Mobile Communications

Lecture 25

OFDM Impairments - Frequency Offset, ISI

Carrier Frequency Offset

- **Carrier frequency offset also causes interchannel interference (ICI).**
- **The transmitted complex envelope with carrier frequency offset Δf is**

$$\tilde{s}(t) = A \sum_{n=0}^{N-1} x_n \exp\left\{ j2\pi \left(\frac{n}{NT_s} + \Delta f \right) t \right\} u_T(t)$$

- **The corresponding IFFT coefficients are**

$$X_k = A \sum_{n=0}^{N-1} x_n \exp\left\{ j2\pi \left(\frac{nk}{N} + k\Delta f T_s \right) \right\} \ , \quad k = 0,\ 1,\ \ldots,\ N+G-1$$

- **The demodulated sequence (in absence of noise) can be written as**

$$Z_l = \mathrm{FFT}\{X_n\} = \eta_l x_l + c_l$$

where

$$\eta_l = A \left\{ \frac{\sin\left(\pi N T_s \Delta f\right)}{\pi N T_s \Delta f} \right\} e^{j\pi N T_s \Delta f}$$

and

$$c_l = A \sum_{n \neq l} a_n H(n,l)$$

is the random ICI term, where

$$H(n,l) = \left\{ \frac{\sin\left[\pi\left(n - l - NT_s\Delta f\right)\right]}{\pi\left[n - l - NT_s\Delta f\right]} \right\} e^{j\pi(n-l-NT_s\Delta f)}$$

Effect of Carrier Frequency Offset

- **Assume that the receiver can determine $\pi N T_s \Delta f = \arg(\eta_l)$, the demodulated sequence in the absence of noise can be written as**

  $$\hat{Z}_l = Z_l e^{-j\pi N T_s \Delta f} = \hat{\eta}_l x_l + \hat{c}_l$$

  where

  $$\hat{\eta}_l = A \left\{ \frac{\sin\left(\pi N T_s \Delta f\right)}{\pi N T_s \Delta f} \right\}$$

  and

  $$\hat{c}_l = A \sum_{n \neq l} a_n \hat{H}(n,l)$$

  is the random ICI term, where

  $$\hat{H}(n,l) = \left\{ \frac{\sin\left[\pi\left(n - l - \sin N T_s \Delta f\right)\right]}{\pi\left(n - l - N T_s \Delta f\right)} \right\} e^{j\pi(n-l-1-N T_s \Delta f)}$$

- **Observe that carrier frequency offset has two effects**

  1. **It reduces the useful signal energy by the factor**

     $$10\log_{10}\left\{ \frac{\sin\left(\pi N T_s \Delta f\right)}{\pi N T_s \Delta f} \right\}^2$$

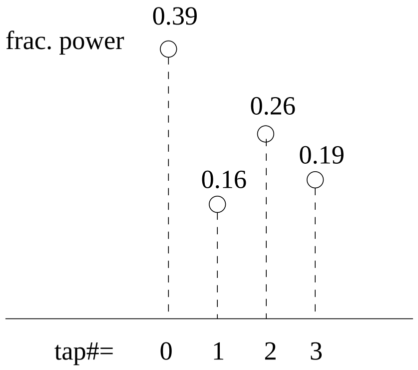
  2. **Introduces an additional additive noise term $\hat{c}_l$**

Residual Intersymbol Interference

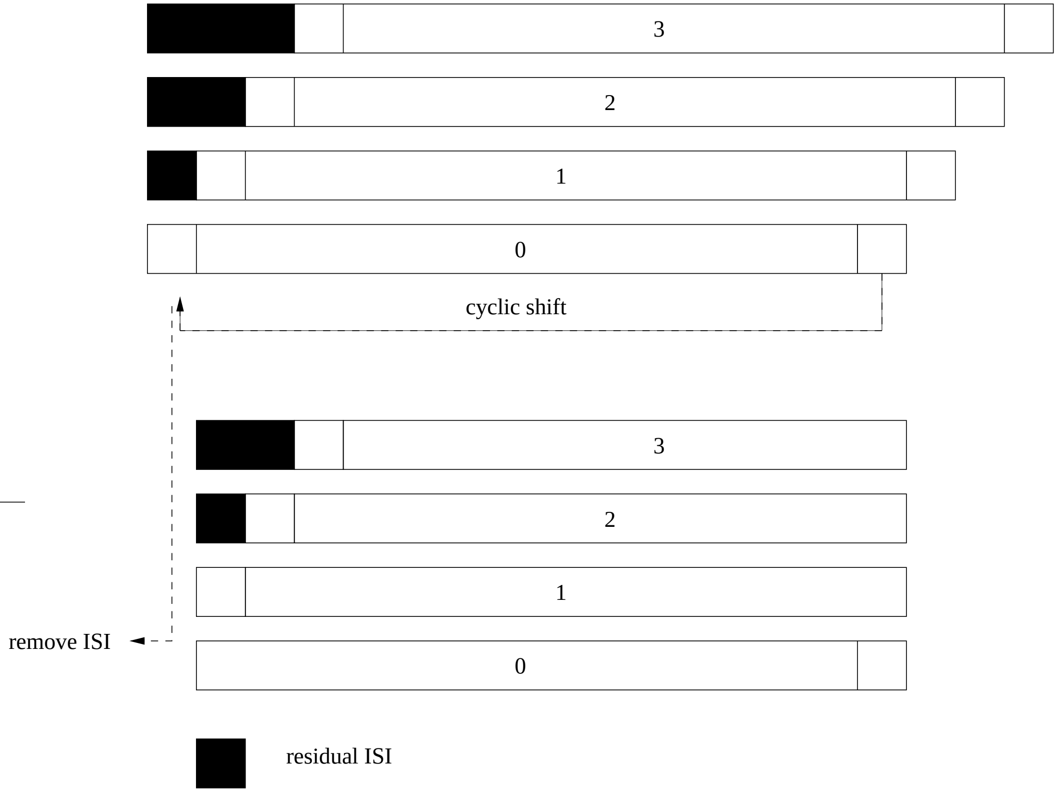
- **With OFDM, ISI is easily mitigated by using a guard interval.**
- **However, when duration of ISI is longer than guard interval, orthogonality between subchannels is no longer maintained and interchannel interference (ICI) arises.**
- **The portion of ISI that exceeds guard interval is called residual ISI.**
- **Residual ISI can degrade performance of OFDM system by introducing high error floors.**
- **Solution is to use time or frequency domain equalization techniques.**

Mitigation of ISI Using Guard Interval

- Assume a guard interval of length $G = 1$ and a channel of length $L = 3$. In this case, there will be residual ISI.



Ch. 2



Intersymbol Interference (ISI)

- **In the absence of noise, the received samples for ith block after removal of the guard interval are**

$$R_{i,k} = \sum_{m=0}^{M} h_{m,k} x_{k-m}, \qquad 0 \le k \le N-1 \tag{1}$$

where $h_{m,k}$ is the channel impulse response at lag m and instant k.

- **The received samples are the sum of two components, viz.,**

$$R_{i,k} = R_{i|i-1,k} + R_{i|i,k}$$

where $R_{i|i-1,k}$ is the received sample component with contributions only from block $i-1$ and $R_{i|i,k}$ is the received sample component with contributions only from block i.

- **Then,**

$$\begin{aligned} R_{i|i-1,k} &= \sum_{m=G+1}^{M} h_m x_{i-1,(k-m)_N} (1 - u(k-m+G)) \\ R_{i|i,k} &= \sum_{m=0}^{M} h_m x_{i,(k-m)_N} u(k-m+G) \end{aligned}$$

where $h_{m,k}$ is replaced by h_m since the channel is static, and $u(n)$ is the unit step function.

Intersymbol Interference (ISI)

- **The received sequence $\{R_{i,k}\}_{k=0}^{N-1}$ is demodulated with the N-point FFT**

$$\mathbf{FFT}\{R_{i,k}\} = \mathbf{FFT}\{R_{i|i-1,k}\} + \mathbf{FFT}\{R_{i|i,k}\}. \tag{2}$$

- **We can express $\mathbf{FFT}\{R_{i|i-1,k}\}$ as**

$$\frac{1}{N}\sum_{m=G+1}^{M} h_m \sum_{l=0}^{N-1} x_{i-1,l} \exp\left\{-j\frac{2\pi lm}{N}\right\} \sum_{k=0}^{N-1} u(m-k-G-1)\exp(j\frac{2\pi(l-n)k}{N}) \tag{3}$$

and $\mathbf{FFT}\{R_{i|i,k}\}$ by

$$\begin{aligned} &x_{i,n}\left\{\Sigma_{m=0}^{G} h_m \exp\left\{-j\tfrac{2\pi nm}{N}\right\} + \Sigma_{m=G+1}^{M} h_m \exp\left\{-j\tfrac{2\pi nm}{N}\right\}(1+\tfrac{G}{N}-\tfrac{m}{N})\right\} - \\ &\tfrac{1}{N}\Sigma_{m=G+1}^{M} h_m \Sigma_{\substack{l=0\\ l\neq n}}^{N-1} x_{i,l} \exp\left\{-j\tfrac{2\pi lm}{N}\right\} \Sigma_{k=0}^{N-1} u(m-k-G-1)\exp\left\{j\tfrac{2\pi(l-n)k}{N}\right\}. \end{aligned} \tag{4}$$

- **For symbol n of block i, (3) is the ISI contribution from block $i-1$, the top half of (4) is the useful signal term, and the bottom half of (4) is the ICI term.**

- **We can express (2) as**

$$Z_n \;=\; \mathbf{FFT}\{R_{i,k}\} = \eta_n x_n + i_n$$

where

$$\begin{aligned} \eta_n &= \sum_{m=0}^{G} h_m \exp\left\{-j\frac{2\pi nm}{N}\right\} + \sum_{m=G+1}^{M} h_m \exp\left\{-j\frac{2\pi nm}{N}\right\}\left(1+\frac{G}{N}-\frac{m}{N}\right) \\ i_n &= i_{i,n} + i_{i-1,n} \end{aligned}$$

and where $i_{i,n}$ the ICI term and $i_{i-1,n}$ the ISI term.

Intersymbol Interference (ISI)

- **Next we find the signal-to-interference ratio (SIR) for symbol n defined by**

$$\mathbf{SIR}(n) = E_u(n)/E_i(n) \tag{5}$$

where the useful signal energy is

$$E_u(n) = \frac{1}{2}\mathrm{E}\left[|\eta_n x_n|^2\right] \quad . \tag{6}$$

- **Since the input symbols $\{x_n\}$ are assumed independent, x_n and i_n are also independent. Furthermore, $i_{i,n}$ and $i_{i-1,n}$ are independent as well. Then, interference energy is**

$$\begin{aligned} E_i(n) &= \frac{1}{2}\mathrm{E}[|i_n|^2] \\ &= \frac{1}{2}\mathrm{E}[|i_{i-1,n}|^2] + \frac{1}{2}\mathrm{E}[|i_{i,n}|^2] \\ &= E_{ISI}(n) + E_{ICI}(n) \quad . \end{aligned} \tag{7}$$

- **The useful signal energy in (6) can be expressed as**

$$\begin{aligned} E_u(n) &= E_s \left| \sum_{m=0}^{G} h_m \exp\left\{-j\frac{2\pi nm}{N}\right\} + \sum_{m=G+1}^{M} h_m \exp\left\{-j\frac{2\pi nm}{N}\right\}(1 + \frac{G}{N} - \frac{m}{N}) \right|^2 \\ &\approx E_s \left| \sum_{m=0}^{M} h_m \exp\left\{-j\frac{2\pi nm}{N}\right\} \right|^2, \qquad \textbf{if } M - G << N \quad . \end{aligned}$$

Intersymbol Interference (ISI)

- **From (7),**

$$\begin{aligned} E_{ISI}(n) &= \frac{E_s}{N}\left\{\sum_{m=G+1}^{M}\sum_{m'=m}^{M} 2(m-G)\mathbf{Re}\left[h_m h_m^* \exp\left\{-j\frac{2\pi n(m-m')}{N}\right\}\right]\right. \\ &\quad \left. - \sum_{m=G+1}^{M}|h_m|^2(m-G)\right\} \qquad (8) \\ E_{ICI}(n) &= E_{ISI}(n) - \frac{E_s}{N^2}\left|\sum_{m=G+1}^{M} h_m \exp\left\{-j\frac{2\pi nm}{N}(m-G)\right\}\right|^2 \quad . \qquad (9) \end{aligned}$$

- **Note that the second term in (9) is relatively small when $M - G << N$, in which case $E_I(n) \approx 2E_{ISI}(n)$.**
- **The symbol error rate (SER) for 16-QAM is**

$$\mathbf{SER} = 3Q\left(\sqrt{\frac{1}{5}\gamma_s}\right)\left[1 - \frac{3}{4}Q\left(\sqrt{\frac{1}{5}\gamma_s}\right)\right], \qquad (10)$$

  where γ_s is the received SNR per symbol.

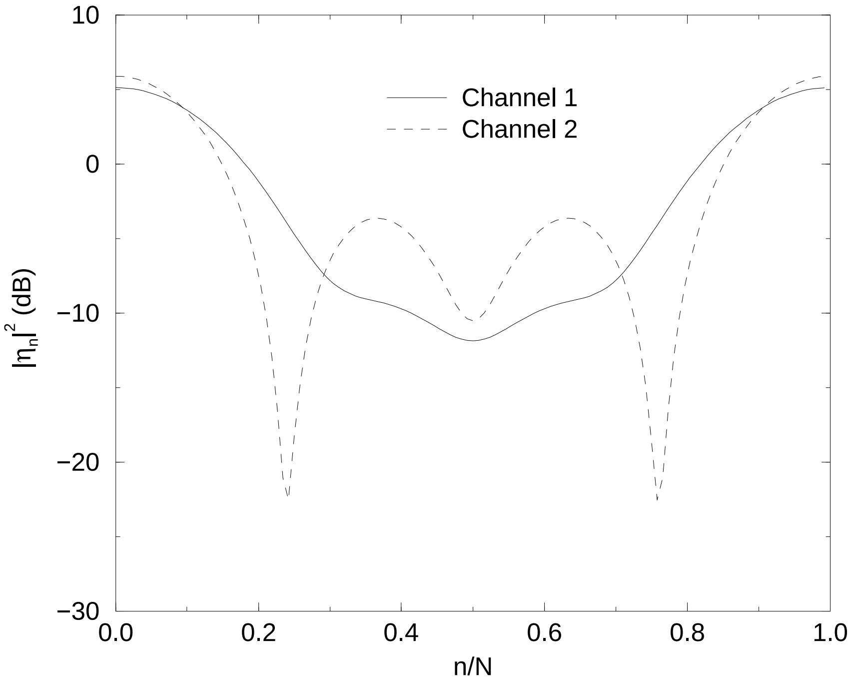
- **With OFDM, the SER on static ISI channel is**

$$\mathbf{SER} = \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{SER}(n)$$

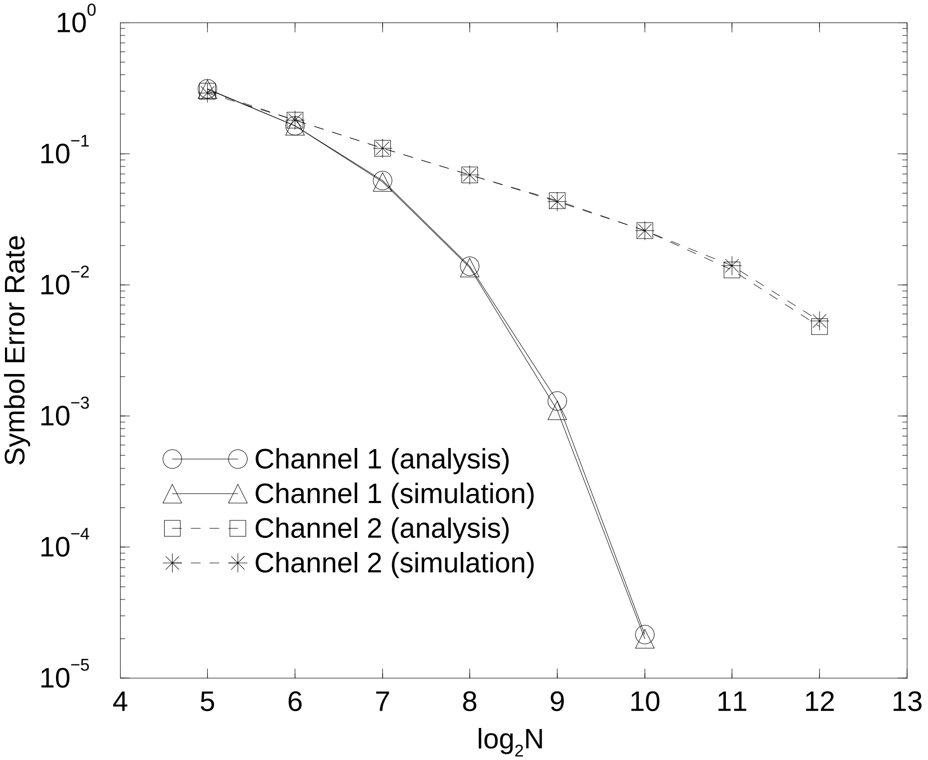
  where $\mathbf{SER}(n)$ is obtained from (10) with γ_s replaced by $\mathbf{SIR}(n)$ from (5).

Static Channels ($N = 128$)

tap #	delay (μs)	Frac. Power (Ch. 1)	Frac. Power (Ch. 2)
0	**0.0**	**0.15**	**0.39**
1	**0.2**	**0.65**	**0.16**
2	**0.4**	**0.15**	**0.26**
3	**0.6**	**0.05**	**0.19**



Uncoded 16-QAM on Static Channels ($G = 0$); 20 Mbps



- **The error floor due to the ISI decreases with increasing block size N. However, when there exists deep null subchannels as in Channel 2, the improvement from increasing block size is quite small. Therefore, increasing block size is not always an effective countermeasure against ISI.**

Fading ISI Channels)

- **On fading ISI channels, channel time variations during a block causes ICI. For large block sizes, the central limit theorem can be invoked and the ICI can be treated like AWGN.**
- **If we assume 2-D isotropic scattering and Rayleigh fading, then for large N the SIR is**

$$\mathbf{SIR} = \frac{E_u}{E_s - E_u} \tag{11}$$

where E_u is

$$\begin{aligned} E_u &= \frac{E_s}{N^2}\left\{\sum_{m=0}^{G} \mathbf{E}|h_m|^2 (N + 2\sum_{i=1}^{N-1}(N-i)J_0(2\pi f_D T_s i)) \right. \\ &\quad \left. + \sum_{m=G+1}^{M} \mathbf{E}|h_m|^2 (N - m + G + 2\sum_{i=1}^{N-m-1+G}(N-m-i+G)J_0(2\pi f_D T_s i))\right\} . \end{aligned}$$

- **The SER for a Rayleigh fading channel is obtained by averaging (10) over the probability density function**

$$p(\gamma_s) = \frac{1}{\bar{\gamma}_s}\exp\left\{-\frac{\gamma_s}{\bar{\gamma}_s}\right\}, \;\; \gamma_s > 0,$$

where $\bar{\gamma}_s$ is replaced by the SIR.

Fading ISI Channels)

- **The SER obtained by using the SIR in (11) is an upper bound when the interference caused by ISI is a dominant factor.**
- **If the channel varies slowly, then we can assume the channel impulse response is constant over a duration of a block. Hence, from (5), the conditional SIR for the subchannel n given the channel impulse response $\underline{h}$, $\mathbf{SIR}(n)_{\underline{h}}$, is**

$$\mathbf{SIR}(n)_{\underline{h}} = \frac{\mathbf{E}[|\eta_n X_n|^2|\underline{h}]}{\mathbf{E}[|I_n|^2|\underline{h}]} \quad .$$

- **For $M - G << N$ $\mathbf{SIR}(n)_{\underline{h}}$ is well approximated by**

$$\begin{aligned}\mathbf{SIR}(n)_{\underline{h}} \;=\; & \frac{E_s}{2}\Bigg\{\frac{\sum_{m=0}^{G}\sum_{m'=0}^{G} h_m h_{m'}^* \exp\left\{-j\frac{2\pi n}{N}(m-m')\right\}}{E_{ISI}(n)_{\underline{h}}} \\ & +\frac{\sum_{m=0}^{G}\sum_{m'=G+1}^{M} h_m h_{m'}^* \exp\left\{-j\frac{2\pi n}{N}(m-m')\right\}}{E_{ISI}(n)_{\underline{h}}} \\ & +\frac{\sum_{m=G+1}^{M}\sum_{m'=G+1}^{M} h_m h_{m'}^* \exp(-j\frac{2\pi n}{N}(m-m'))}{E_{ISI}(n)_{\underline{h}}}\Bigg\} \end{aligned} \qquad (12)$$

where $E_{ISI}(n)_{\underline{h}}$ is from (8) but expressed differently as

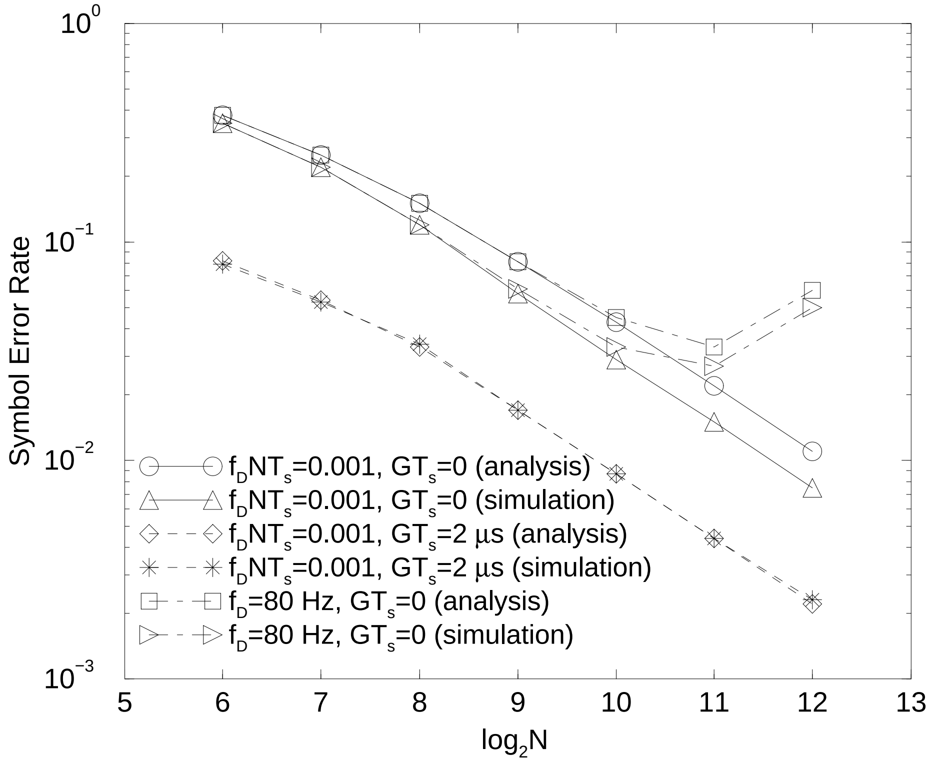
$$E_{ISI}(n)_{\underline{h}} = \frac{E_s}{N}\sum_{m=G+1}^{M}\sum_{m'=G+1}^{M} h_m h_{m'}^* \mathbf{min}[m-G, m'-G]\exp\left\{-j\frac{2\pi n(m-m')}{N}\right\}$$

Fading ISI Channels

- In (12), the first and second fractions represent the portion of SIR for which the Rayleigh assumption is valid in computing the SER, because the useful signal term and the interference term are uncorrelated.
- However, the last fraction in (12) shows that the useful signal term and the interference term are correlated. Hence, when the useful signal term is faded so is the interference. Consequently, the Rayleigh assumption leads to pessimistic performance estimates.
- On the other hand, if the portion of the energy contained within the guard interval is relatively large, then the last fraction becomes insignificant relative to the first two fractions in (12) and, hence, the SER found by using the SIR from (11) is accurate.

16-QAM on Fading Channels

20 Mbps, COST207 RTU Model



- **The block size must be small enough to keep the ICI small, while it must be large enough to keep the ISI small for channels with a long impulse response.**