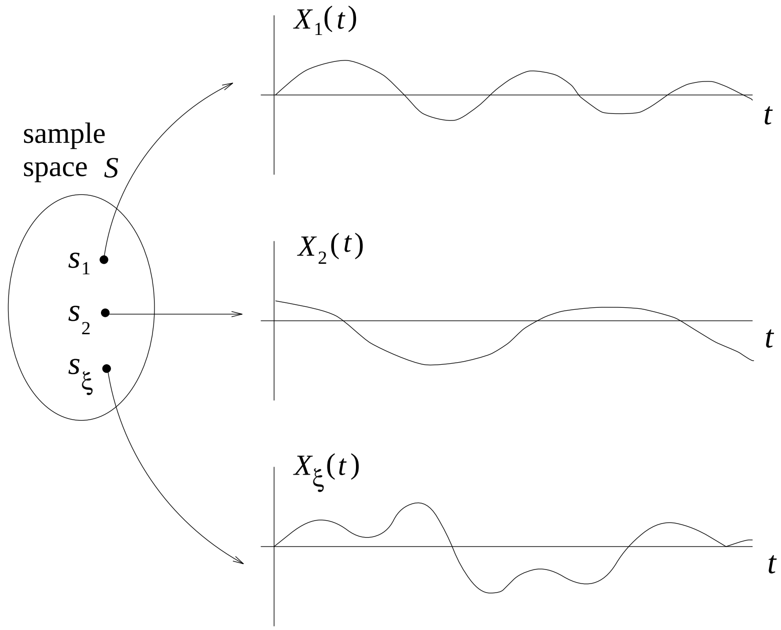
ECE6604
PERSONAL & MOBILE COMMUNICATIONS

Lecture 6

Envelope Correlation



Ensemble of Sample Functions for a Random Process.

Autocorrelation Functions

- **Let $X(t)$ denote a random process.**
- **At any time t, $X(t)$ is a random variable with probability density function $f_{X(t)}(x)$.**
- **The *ensemble* mean of $X(t)$ is**

$$\mu_X(t) = \mathrm{E}[X(t)] = \int x f_{X(t)}(x) dx$$

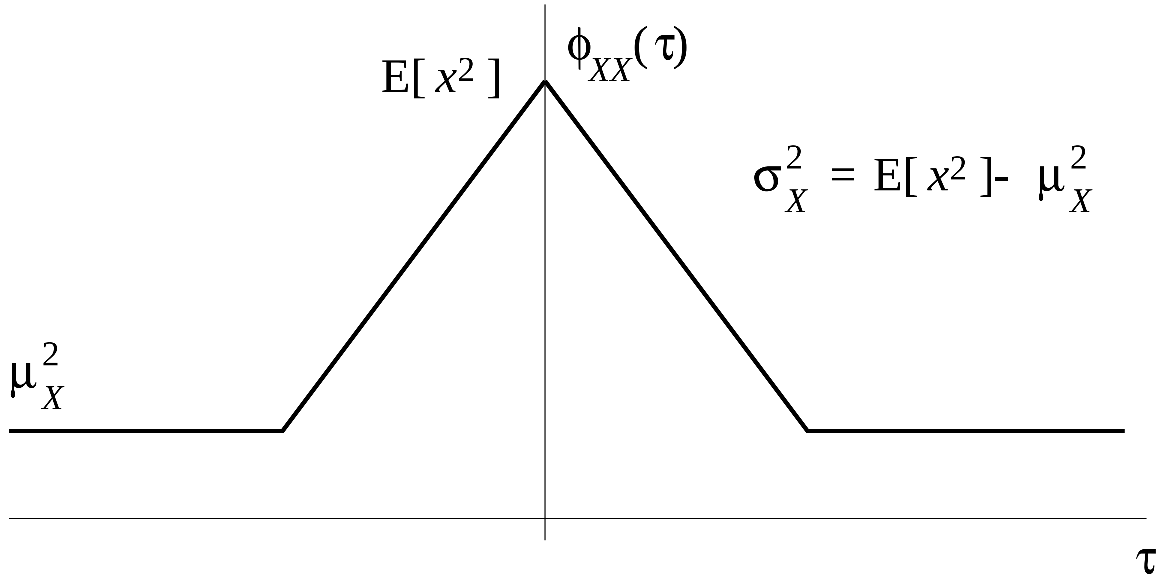
- **The autocorrelation of $X(t)$ is**

$$\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)] = \int \int x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2) dx_1 dx_2$$

- **For a *wide sense stationary* random process**

$$\begin{aligned} \mu_X(t) &= \mu_X \\ \phi_{XX}(t_1, t_2) &= \phi_{XX}(t_2 - t_1) = \phi_{XX}(\tau) \end{aligned}$$

where $\tau = t_2 - t_1$.



A typical autocorrelation function for a random process.

Autocorrelation of the Bandpass Signal

- **Consider again the band-pass random process**

$$r(t) = g_I(t)\cos 2\pi f_c t \;-\; g_Q(t)\sin 2\pi f_c t$$

where

$$\begin{aligned} g_I(t) &= \sum_{n=1}^{N} C_n \cos\phi_n(t) \\ g_Q(t) &= \sum_{n=1}^{N} C_n \sin\phi_n(t) \end{aligned}$$

- **Assuming that $r(t)$ is wide-sense stationary, the autocorrelation of $r(t)$ is**

$$\begin{aligned} \phi_{rr}(\tau) &= \mathrm{E}[r(t)r(t+\tau)] \\ &= \mathrm{E}[g_I(t)g_I(t+\tau)]\cos 2\pi f_c\tau + \mathrm{E}[g_Q(t)g_I(t+\tau)]\sin 2\pi f_c\tau \qquad \textbf{Typo in Text!} \\ &= \phi_{g_Ig_I}(\tau)\cos 2\pi f_c\tau - \phi_{g_Ig_Q}(\tau)\sin 2\pi f_c\tau \end{aligned}$$

where $\mathrm{E}[\;\cdot\;]$ is the ensemble average operator, and

$$\begin{aligned} \phi_{g_Ig_I}(\tau) &\stackrel{\triangle}{=} \mathrm{E}[g_I(t)g_I(t+\tau)] \\ \phi_{g_Ig_Q}(\tau) &\stackrel{\triangle}{=} \mathrm{E}[g_I(t)g_Q(t+\tau)] \;\; . \end{aligned}$$

Note that the wide-sense stationarity of $r(t)$ imposes the condition (Homework!!!

$$\begin{aligned} \phi_{g_Ig_I}(\tau) &= \phi_{g_Qg_Q}(\tau) \\ \phi_{g_Ig_Q}(\tau) &= -\phi_{g_Qg_I}(\tau) \;\; . \end{aligned}$$

Auto- and Cross-correlation of Quadrature Components

- **The phases $\phi_n(t)$ are statistically independent random variables at any time t, uniformly distributed over the interval $[-\pi, \pi)$.**
- **The azimuth angles of arrival, θ_n are all independent due to the random placement of scatterers. Also, in the limit $N \to \infty$, the discrete azimuth angles of arrival θ_n can be replaced by a continuous random variable θ having the probability density function $p(\theta)$.**
- **By using the above properties, the auto- and cross-correlation functions can be obtained as follows:**

$$\phi_{g_I g_I}(\tau) = \phi_{g_Q g_Q}(\tau) = \lim_{N\to\infty} \mathrm{E}_{\boldsymbol{\tau},\boldsymbol{\theta}}[g_I(t)g_I(t+\tau)] = \frac{\Omega_p}{2}\mathrm{E}_\theta[\cos(2\pi f_m \tau \cos\theta)]$$

$$\phi_{g_I g_Q}(\tau) = -\phi_{g_Q g_I}(\tau) = \lim_{N\to\infty} \mathrm{E}_{\boldsymbol{\tau},\boldsymbol{\theta}}[g_I(t)g_Q(t+\tau)] = \frac{\Omega_p}{2}\mathrm{E}_\theta[\sin(2\pi f_m \tau \cos\theta)]$$

$$\begin{aligned} \boldsymbol{\tau} &= (\tau_1, \tau_2, \ldots, \tau_N) \\ \boldsymbol{\theta} &= (\theta_1, \theta_2, \ldots, \theta_N) \\ \Omega_p &= \mathrm{E}[g_I^2(t)] + \mathrm{E}[g_Q^2(t)] = \sum_{n=1}^{N} C_n^2 \end{aligned}$$

and Ω_p is the total received envelope power.

2-D Isotropic Scattering

- **Evaluation of the expectations for the auto- and cross-correlation functions requires the azimuth distribution of arriving plane waves $p(\theta)$, and the receiver antenna gain pattern $G(\theta)$, as a function of the azimuth angle θ.**
- **With 2-D isotropic scattering, the plane waves are confined to the $x - y$ plane and arrive uniformly distributed angle of incidence, i.e.,**

$$p(\theta) = \frac{1}{2\pi} \quad -\pi \leq \theta \leq \pi$$

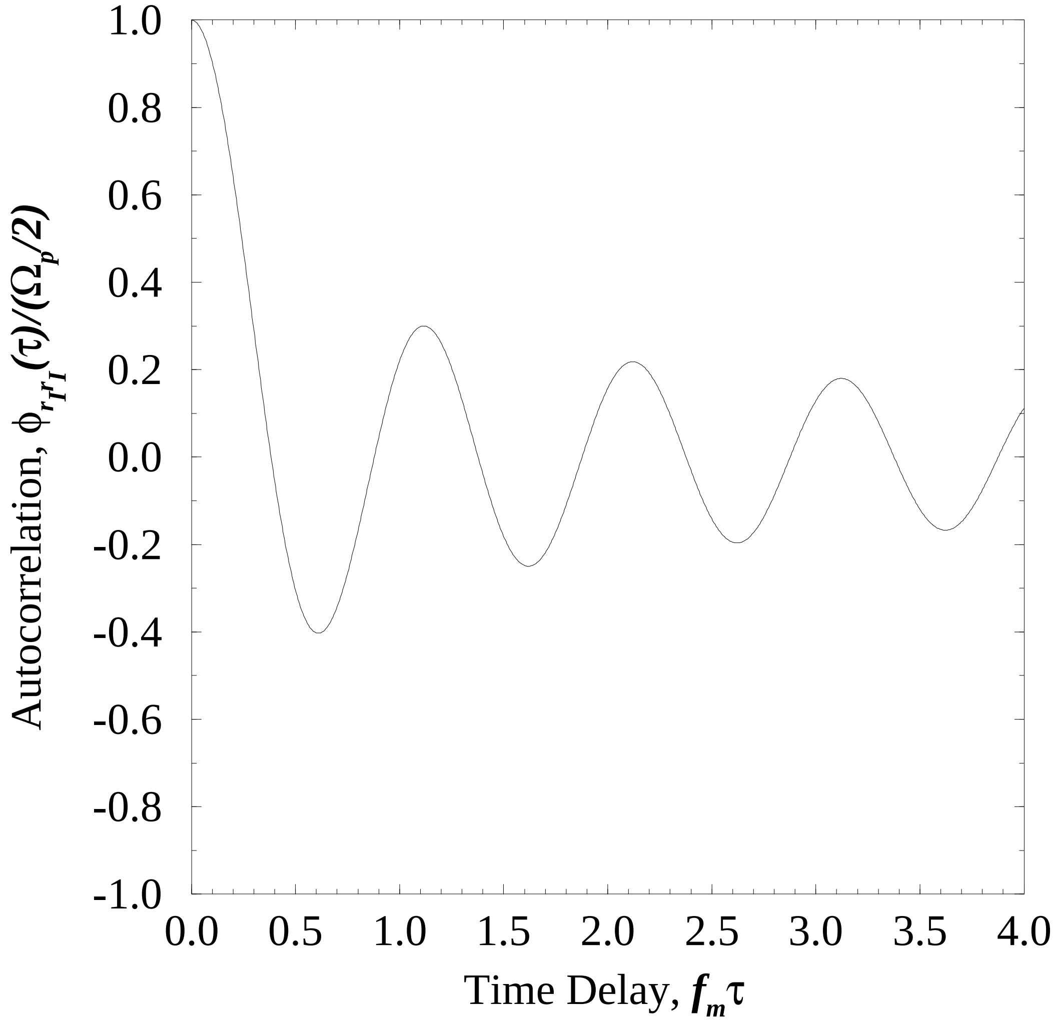
- **With 2-D isotropic scattering and an isotropic receiver antenna with gain $G(\theta) = 1, \theta \in [-\pi, \pi)$, the auto- and cross-correlation functions become**

$$\begin{aligned} \phi_{g_I g_I}(\tau) &= = \frac{\Omega_p}{2} J_0(2\pi f_m \tau) \\ \phi_{g_I g_Q}(\tau) &= 0 \end{aligned}$$

where

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \cos\theta) \mathrm{d}\theta$$

is the zero-order Bessel function of the first kind.



Normalized autocorrelation function of the quadrature components of the received complex envelope with 2-D isotropic scattering and an isotropic receiver antenna.

Doppler Spectrum

- **The autocorrelation function and power spectral density (psd) are Fourier transform pairs.**

$$\begin{aligned} S_{gg}(f) &= \int_{-\infty}^{\infty} \phi_{gg}(\tau) e^{-j2\pi f\tau} d\tau \\ \phi_{gg}(\tau) &= \int_{-\infty}^{\infty} S_{gg}(\tau) e^{j2\pi f\tau} d\tau \end{aligned}$$

- **The autocorrelation of the received complex envelope** $g(t) = g_I(t) + jg_Q(t)$ **is**

$$\begin{aligned} \phi_{gg}(\tau) &= \frac{1}{2}\mathrm{E}[g^*(t)g(t+\tau)] \\ &= \phi_{g_I g_I}(\tau) + j\phi_{g_I g_Q}(\tau) \end{aligned}$$

- **The Fourier transform of** $\phi_{gg}(\tau)$ **gives the Doppler psd**

$$S_{gg}(f) = S_{g_I g_I}(f) + jS_{g_I g_Q}(f) \quad .$$

Sometimes $S_{gg}(f)$ **is just called the "Doppler spectrum."**

Doppler Spectrum

- **We can also relate the power spectrum of the complex envelope $g(t)$ to that of the band-pass process $r(t)$. We have**

$$\phi_{rr}(\tau) = \mathrm{Re}\left[\phi_{gg}(\tau)e^{j2\pi f_c \tau}\right] \quad .$$

- **By using the identity**

$$\mathrm{Re}\,[z] = \frac{z + z^*}{2}$$

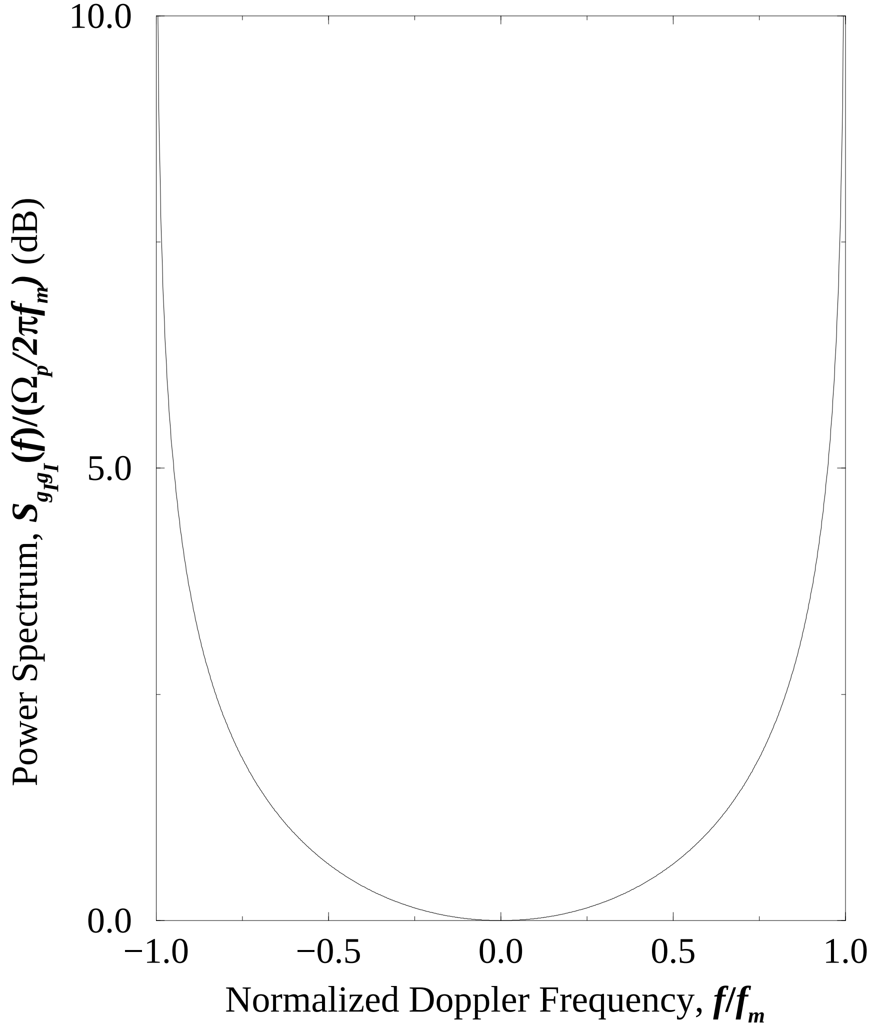
and the property $\phi_{gg}(\tau) = \phi^*_{gg}(-\tau)$, it follows that the band-pass Doppler psd is

$$S_{rr}(f) = \frac{1}{2}\left[S_{gg}(f - f_c) + S_{gg}(-f - f_c)\right] \quad .$$

- **Since $\phi_{gg}(\tau) = \phi^*_{gg}(-\tau)$, the Doppler spectrum $S_{gg}(f)$ is always a real-valued function of frequency, but not necessarily even. However, the band-pass Doppler spectrum $S_{rr}(f)$ is always real-valued and even.**

- **For 2-D isotropic scattering, the psd and cross psd of $g_I(t)$ and $g_Q(t)$ are**

$$\begin{aligned} S_{g_I g_I}(f) &= \mathcal{F}[\phi_{g_I g_I}(\tau)] = \begin{cases} \frac{\Omega_p}{4\pi f_m} \frac{1}{\sqrt{1-\left(\frac{f}{f_m}\right)^2}} & |f| \leq f_m \\ 0 & \textbf{otherwise} \end{cases} \\ S_{g_I g_Q}(f) &= 0 \end{aligned}$$



Normalized psd of the quadrature components of the received complex envelope with 2-D isotropic scattering and an isotropic receiver antenna. Sometimes this is called the CLASSICAL Doppler power spectrum.

Rician Fading

- **Suppose that the propagation environment consisting of a strong specular component plus a scatter component. The azimuth distribution $p(\theta)$ might have the form**

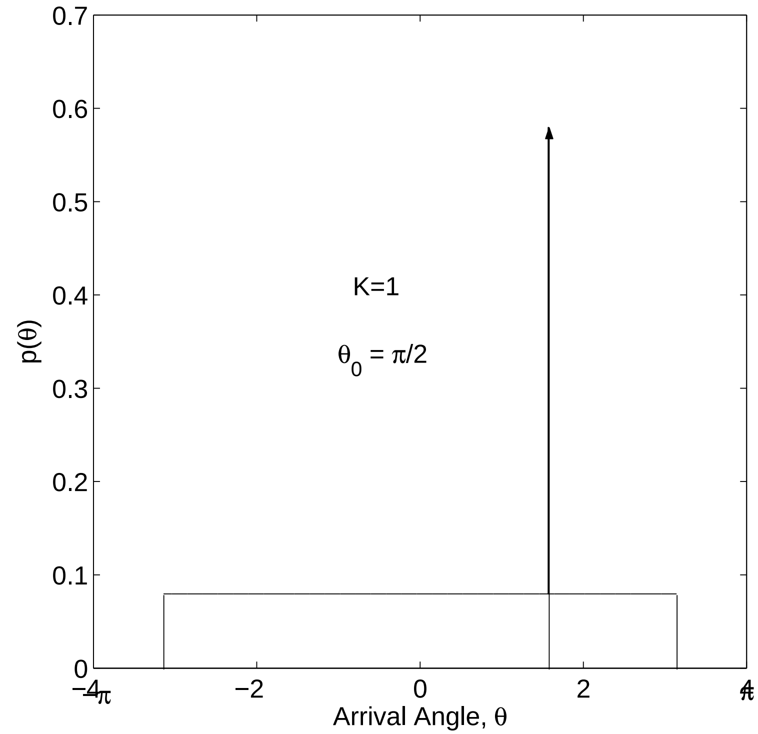
$$p(\theta) = \frac{1}{K+1}\hat{p}(\theta) + \frac{K}{K+1}\delta(\theta - \theta_0)$$

where $\hat{p}(\theta)$ is the continuous AoA distribution of the *scatter* **component, θ_0 is the AoA of the specular component, and K is the ratio of the received specular to scattered power.**

- **One such scattering environment, assumes that the scatter component exhibits 2-D isotropic scattering, i.e., $\hat{p}(\theta) = 1/(2\pi), \theta \in [-\pi, \pi)$.**

- **The correlation functions $\phi_{g_I g_I}(\tau)$ and $\phi_{g_I g_Q}(\tau)$ are**

$$\phi_{g_I g_I}(\tau) = \frac{1}{K+1}\frac{\Omega_p}{2}J_0(2\pi f_m \tau) + \frac{K}{K+1}\frac{\Omega_p}{2}\cos(2\pi f_m \tau \cos\theta_0)$$
$$\phi_{g_I g_Q}(\tau) = \frac{K}{K+1}\frac{\Omega_p}{2}\sin(2\pi f_m \tau \cos\theta_0) \quad .$$



Plot of $p(\theta)$ *vs.* θ *with 2-D isotropic scattering plus a LoS or specular component arriving at angle* $\theta_0 = \pi/2$.

- **The azimuth distribution**

$$p(\theta) = \frac{1}{K+1}\hat{p}(\theta) + \frac{K}{K+1}\delta(\theta - \theta_0)$$

yields a complex envelope having a Doppler spectrum of the form

$$S_{gg}(f) = \frac{1}{K+1}S_{gg}^c(f) + \frac{K}{K+1}S_{gg}^d(f) \qquad (1)$$

where $S_{gg}^d(f)$ **is the discrete portion of the Doppler spectrum due to the specular component and** $S_{gg}^c(f)$ **is the continuous portion of the Doppler spectrum due to the scatter component.**

- **For the case when** $\hat{p}(\theta) = 1/(2\pi), \theta \in [-\pi, \pi]$**, the power spectrum of** $g(t) = g_I(t) + jg_Q(t)$ **is**

$$S_{gg}(f) = \begin{cases} \frac{1}{K+1} \cdot \frac{\Omega_p}{2\pi f_m} \frac{1}{\sqrt{1-(f/f_m)^2}} \\ \qquad + \frac{K}{K+1}\frac{\Omega_p}{2}\delta(f - f_m \cos\theta_0) & 0 \leq |f| \leq f_m \\ 0 & \textbf{otherwise} \end{cases} .$$

- **Note the discrete tone at frequency** $f_c + f_m \cos\theta_0$ **due to the line-of-sight or specular component arriving from angle** θ_0**.**

Non-isotropic scattering

- **Sometimes the azimuth distribution $p(\theta)$ may not be uniform, a condition commonly called non-isotropic scattering. Several distributions have been suggested to model non-isotropic scattering.**

- **Once possibility is the Gaussian distribution**

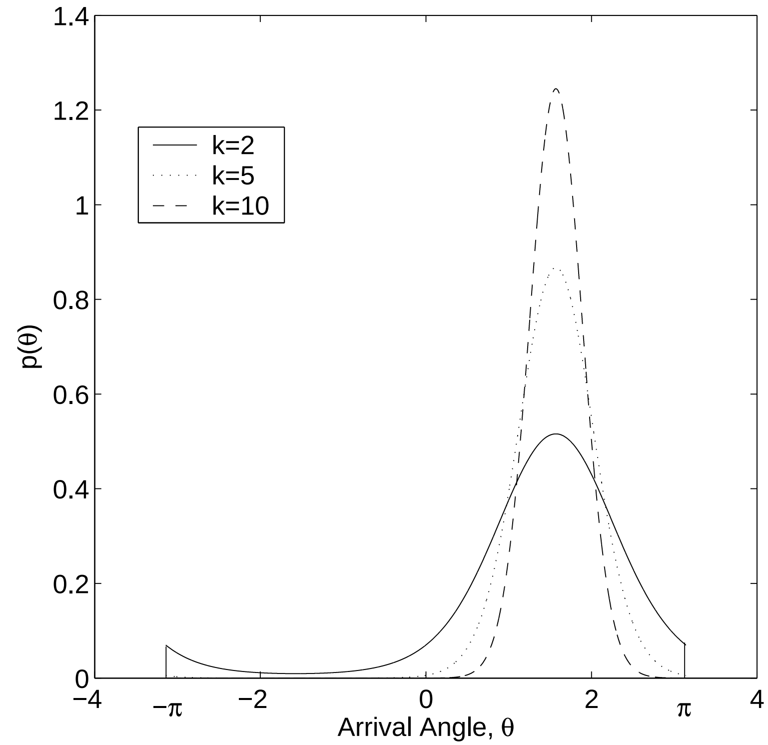
$$p(\theta) = \frac{1}{\sqrt{2\pi}\sigma_S} \exp\left\{-\frac{(\theta-\mu)^2}{2\sigma_S^2}\right\}$$

  where μ is the mean AoA, and σ_S is the rms AoA spread.

- **Another possibility is the von Mises distribution**

$$p(\theta) = \frac{1}{2\pi I_0(k)} \exp\left[k\cos(\theta-\mu)\right] \ ,$$

  where $\theta \in [-\pi, \pi)$, $I_0(\ \cdot\ )$ is the zeroth-order modified Bessel function of the first kind, $\mu \in [-\pi, \pi)$ is the mean AoA, and k controls the spread of scatterers around the mean.



Plot of $p(\theta)$ vs. θ for the von Mises distribution with a mean angle-to-arrival $\mu = \pi/2$.