Bit-Allocation Strategies for MIMO Fading Channels with Channel Knowledge at Transmitter

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***Abstract* — The singular-value decomposition can be used to transform a MIMO fading channel into an equivalent bank of scalar subchannels, also known as eigenbeamforming or closed-loop MIMO, provided that the transmitter knows the channel. We consider the problem of allocating bits to subchannels after such processing, and propose simple strategies with near-optimal performance exploiting properties of singular values. For large antenna arrays, a fixed bit-allocation becomes an attractive choice without any significant performance loss. For example, on the 6-input 6-output Rayleigh fading, the fixed allocation strategy performs only 0.25 dB worse than the optimal bit-allocation in terms of required SNR.**

I. INTRODUCTION

For a bank of scalar channels, it is well-known that capacity is achieved by water-pouring procedures when channel information is known to the transmitter [1]. In practice, where rates are often restricted to be finite and discrete values, the problem must be modified, instead of optimal rate by water-pouring, to allocate information rate to parallel channels with granularity constraint.

The best way to allocate discrete rates to parallel channels would compare all possible combinations of allocations and select the best one. However, this exhaustive search often requires high complexity and thus iterative optimization is widely used instead. Many iterative algorithms for allocation have been introduced, especially for discrete multitone (DMT) applications, such as [2]-[4], which has relatively small complexity without significant performance loss.

For MIMO channels, if transmitter knows the channel, parallel subchannels are created by singular-value decomposition (SVD) (See [5] and references therein). Unlike DMT, MIMO flat-fading channels have some unique properties: the number of subchannels in MIMO systems is small compared to DMT, and singular values of channel matrices, which are subchannel gains, show special properties. In this paper, we study allocation problem exploiting properties of MIMO channels.

The rest of this paper is organized as follows. Section II describes MIMO flat-fading model and make problem statement. We propose low-complexity bit-allocation strategies, and explain near-optimal performance of proposed strategies in Section III. Performance is evaluated for Rayleigh fading in Section IV. In Section V, we deal with robustness of proposed strategies to the change of fading statistics. Finally, we conclude in Section VI.

II. SYSTEM MODEL AND BIT-ALLOCATION PROBLEM

For simplicity we consider a narrowband channel with M transmit and M receive antennas, which can be modeled by an $M \times M$ channel matrix $\mathbf{H} = [h_{ij}]$, where h_{ij} is the response at receive antenna i from transmit antenna j. Let $\mathbf{H} = \mathbf{U}\mathrm{diag}(\boldsymbol{s}^{1/2})\mathbf{V}^*$ denote an SVD, where $\mathbf{U}$ and $\mathbf{V}$ are unitary, and where the elements of $\boldsymbol{s} = [s_1, \ldots, s_M]$ are real and ordered so that $s_1 \geq \ldots \geq s_M \geq 0$. When the transmitter and receiver filter by $\mathbf{V}$ and $\mathbf{U}^*$, respectively, a bank of *scalar* subchannels results:

$$y_i = \sqrt{s_i}\, a_i + n_i\,, \quad \text{for } i = 1, \ldots M\,, \tag{1}$$

where $\{n_i\}$ are *i.i.d.* $\mathcal{CN}(0, N_0)$. There is no crosstalk from one subchannel to the next.

So as to achieve a rate of r_i bits per signaling interval across the i-th subchannel, its SNR $s_i E_i / N_0$ must be at least $\Gamma(2^{r_i} - 1)$, where $E_i = \mathrm{E}[\,|a_i|^2]$, and where Γ, an SNR gap, accounts for additional SNR required for any practical code to achieve a given target probability of error instead of using an ideal capacity-achieving code [2]. Then, given channels and an SNR gap, the total energy, $\sum_i E_i$, required by the transmitter to achieve a given set of rates $\{r_i\}$ is:

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$$E(\boldsymbol{s}) = \Gamma\sum_{i=1}^{M}\frac{2^{r_i}-1}{s_i/N_0} \quad . \qquad (2)$$

It is well-known that, to achieve a given total rate of $R = \sum_i r_i$ bits per signaling interval, the rate allocation that minimizes (2) is given by the water-pouring solution, $r_i = \{\log_2(\lambda(\boldsymbol{s})s_i/\Gamma)\}^+$, where $\{x\}^+ = \max\{0, x\}$, and where $\lambda(\boldsymbol{s})$ ensures that $R = \sum_i r_i$.

In practice, complexity considerations require that $\{r_i\}$ be drawn from a *discrete* and *finite* set. Let the granularity, β, be the smallest incremental unit of information rate. Then, the rate of any subchannel is given by $r_i = \beta B_i$, where B_i is a non-negative integer. With these constraints, the *bit-allocation problem* is to find the $\{r_i\}$, given $\boldsymbol{s}$, that minimizes (2) subject to a total rate constraint, $R = \sum_i r_i$. Clearly, the best bit-allocation is based on a full search that enumerates each element of:

$$\mathcal{B} = \{[r_1, \ldots, r_M]; \sum_i r_i = \beta\sum_i B_i = R, r_1 \geq \ldots \geq r_M \geq 0\} \quad ,(3)$$

and chooses the allocation that minimizes (2) for given $\boldsymbol{s}$. The ordering restriction on $\{r_i\}$ in (3) stems from the ordered nature of $\boldsymbol{s}$. When M is large, such as in DMT, the size of full-search set ($\mathcal{B}$) can be very large and calculating (2) for all members in $\mathcal{B}$ might be practically too complex. Even for MIMO flat-fading channels, where the number of subchannels is not as large as DMT due to physical space limitation of antenna arrays, the complexity of full-search strategy can be high. Usually, full-search strategy seems to be feasible only for $M = 2$. In the paper, we consider how to reduce the complexity without any significant performance loss.

III. Proposed Bit-Allocation Strategy

First we investigate how frequently an allocation in the full-search set is used and how it contributes to average required SNR, when channels, $\{h_{ij}\}$, are generated according to a certain distribution, such as Rayleigh distribution. Let $\boldsymbol{b}_j = [b_{1j}, \ldots, b_{Mj}]$ be the j-th allocation in $\mathcal{B}$ and let A_j be a subregion in M-dimensional space, $\{\boldsymbol{s}; s_1 \geq \ldots \geq s_M \geq 0\}$, in which $\boldsymbol{b}_j$ is optimal. In other words, if $\boldsymbol{s} \in A_j$, $\boldsymbol{b}_j$ is the bit-allocation that minimizes (2) with $r_i = b_{ij}$ for $i = 1, \ldots M$. Then, average required SNR for full search becomes:

$$E/N_0 = \mathrm{E}_{A_j}\left[\mathrm{E}_{\boldsymbol{s}}\left[\Gamma\sum_{i=1}^{M}\frac{2^{r_i}-1}{s_i}\mid A_j\right]\right] = \sum_{j=1}^{L}P_j\varepsilon_j \quad , (4)$$

where $\varepsilon_j = \Gamma\sum_i(2^{R_c b_{ij}}-1)\mathrm{E}_{\boldsymbol{s}}[1/s_i|A_j]$ is the partial SNR requirement conditioned on A_j, and where L denotes the size of $\mathcal{B}$. The probability mass function (PMF) of allocation is denoted by $P_j = \mathrm{Prob}[A_j]$ for $j = 1, \ldots L$, which indicates how often $\boldsymbol{b}_j$ is selected over realizations of $\boldsymbol{s}$.

One way to reduce search-set size is based on the following observations. For given fading statistics, some elements of $\mathcal{B}$ are infrequently or never used, that is, P_j is very small or zero for some j.

> **Observation 1.** If P_j is nominal, deleting its allocation $\boldsymbol{b}_j$ from $\mathcal{B}$ and using other allocation(s) for A_j will increase E/N_0, but its increase is only marginal.

This is obvious since small P_j nulls the increase in $P_j\varepsilon_j$ by using suboptimal allocation for A_j. Hence, deleting these infrequent allocations from consideration has little impact on performance.

In order to see how deleting members from $\mathcal{B}$ impacts performance, we investigate increase in average SNR by deleting members one by one. We delete the allocation that has the smallest P_j, and calculate corresponding average SNR penalty compared to full-search strategy. Repeat these procedures until all but one allocation are eliminated. Fig. 1-a illustrates average SNR penalty in dB as allocations are removed from $\mathcal{B}$ for $\beta = 3/4$ and $B = 12$, where we assume $B_i \in \{0, \ldots 8\}$ in (3) and, where 10^5 independent Rayleigh channels ($M = 6$) are generated. As illustrated in Fig. 1-b, which plots P_j for $j = 1, \ldots L$, there are seven allocations out of $L = 51$ which have dominant P_j. Labels in Fig. 1-b identify seven dominant-P_j allocations, and impact of eliminating them is shown in Fig. 1-a, where alphabetical order matches deletion order. It can be seen in Fig. 1-a that penalty is almost zero until seven labeled allocations are left in reduced set, which agrees with Observation 1. By removing the 'a' allocation, SNR penalty begins to grow sharply, and the last elimination (labeled as 'f') leaves only one allocation (labeled as 'survivor') in the search set.

An interesting point in this elimination process is how many allocations have nominal P_j, so that they do not affect (4) much. As observed in Fig. 1-b, only a few out of L possibilities have dominant P_j, and thus the number of considerations reduces correspondingly.

> **Observation 2.** In MIMO channels the number of allocations with dominant P_j is small relatively to the size of full-search set, L.

This can be explained in part by ordered natures of singular values and reduced variability of bit-allocation.

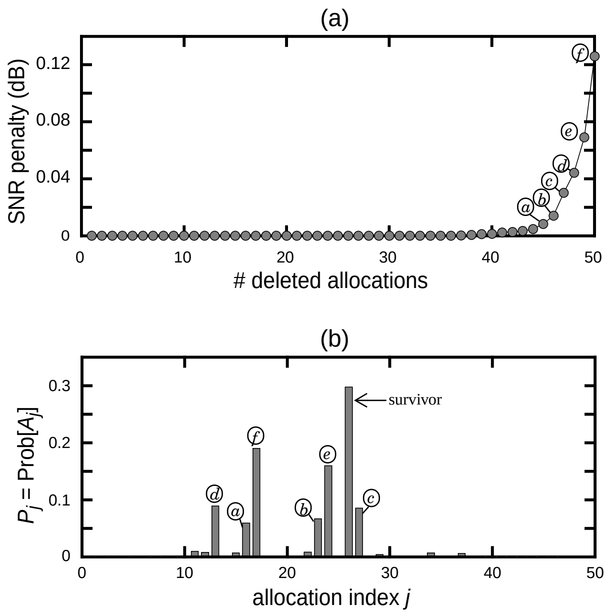

Fig. 1. (a) SNR penalty in dB due to deleting bit-allocations from $\mathcal{B}$, (b) PMF (P_j) of bit-allocations, both for $M = 6$ and $B = 12$.

By ordered nature, we mean not only that singular values are ordered, $s_1 \geq \ldots \geq s_M$, but also that each singular value has a different distribution. Thus, depending on distributions of singular values, some allocations have higher probabilities than others, which implies that bit-allocation is more predictable. Certainly, there is still variability in bit-allocation despite of ordered nature. The point is how small this variability is. To this purpose, we investigate distribution of optimal rate, $r_i = \{\log_2(\lambda(\boldsymbol{s})s_i / \Gamma)\}^+$. For instance, Fig. 2 illustrates empirical marginal distribution of optimal rate for $M = 4$ (thick) and $M = 6$ (thin) at $R / M = 2$ bits per signaling interval and per antenna, where we assume $\mathrm{E}[\,|h_{ij}|^2] = 1$. Even though marginal distribution tells only a part of the whole story, a variability reduction is obvious as one goes from $M = 4$ to $M = 6$ in Fig. 2. In M-dimensional space, we conjecture, from marginal distribution and covariance of optimal rates, that only a small portion of space corresponds to large probability as joint distribution is centered and has small variance (in all directions), and that distribution shrinks, that is, variance becomes small for large M. This conjecture suggests that only a few allocations in $\mathcal{B}$, whose A_j corresponds to high-probability regions of optimal rates, have dominant P_j.

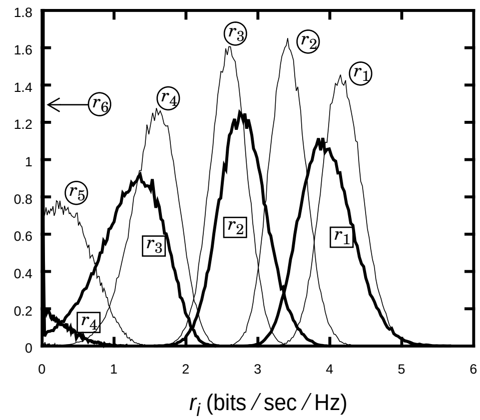

Fig. 2. Marginal PDF of optimal rate in () for $M = 4$ (thick) and for $M = 6$ (thin) at $R / M = 2$ bits / sec / Hz / antennas.

Now we move further and delete some of allocations with dominant P_j, as inspired from Fig. 1-a, where deleting all but one allocation incurs only a penalty of 0.13 dB compared to full search.

Observation 3. A penalty by removing $\boldsymbol{b}_j$ with dominant P_j is not negligible any more, but still reasonably small.

This is based on the fact that the increase in ε_j in (4) is small, even if P_j is not nominal, since distance between the optimal $\boldsymbol{b}_j$ and its substitute is not quite far as regions of frequently-used allocations concentrate in M-dimensional space.

From Observations 1-3, we propose bit-allocation strategies restricting its search to $\mathcal{B}_1$ and $\mathcal{B}_2$, where $\mathcal{B}_k$ denotes a restricted search set containing only k candidate allocations. For optimal choice of $\mathcal{B}_k$, we compare average required SNR for all possible members of $\mathcal{B}_k$, and choose the one that produces minimum average required SNR. For most cases, optimal choice coincides with the results by the results by deleting infrequently-used allocation one by one as in Fig. 1-a, but it is not always true, especially when $M = 2$.

Advantages of proposed strategies include: (i) a great reduction in complexity; (ii) thus suitable to frequent channel change; (iii) no increase in complexity as number of subchannels (M) grows; and (iv) applicable to any constraint on rate (*e.g.* any stepsize β or any maximum value of B_i). Complexity reduction is quite impressive when compared to full search. For example, only two calculations of (2) are required if $\mathcal{B}_2$ is used, in contrast to 51 calculations required for a full search when $M = 6$ and

$M = 2$

	$\mathcal{B}_2$	$\mathcal{B}_1$
$B = 2$	[2 0], [1 1]	[2 0]
$B = 4$	[4 0], [3 1]	[4 0]
$B = 6$	[6 0], [5 1]	[6 0]
$B = 8$	[8 0], [6 2]	[8 0]

$M = 4$

	$\mathcal{B}_2$	$\mathcal{B}_1$
$B = 4$	[3 1 0 0], [2 2 0 0]	[3 1 0 0]
$B = 8$	[4 3 1 0], [5 3 0 0]	[4 3 1 0]
$B = 12$	[5 4 3 0], [6 4 2 0]	[6 4 2 0]
$B = 16$	[7 5 4 0], [7 6 3 0]	[7 6 3 0]

$M = 6$

	$\mathcal{B}_2$	$\mathcal{B}_1$
$B = 4$	[3 3 2 0 0 0], [4 3 1 0 0 0]	[4 3 1 0 0 0]
$B = 12$	[5 4 2 1 0 0], [4 4 3 1 0 0]	[5 4 2 1 0 0]
$B = 18$	[6 5 3 2 0 0], [5 5 4 2 0 0]	[6 5 3 2 0 0]
$B = 24$	[6 5 4 3 2 0], [7 5 4 3 1 0]	[7 5 4 3 1 0]

Fig. 3. Samples of restricted sets ($\mathcal{B}_1$ and $\mathcal{B}_2$) optimized to Rayleigh for $M \in \{2, 4, 6\}$.

$B = 12$ for $B_i \in \{0, \ldots 8\}$. Especially restricting search to $\mathcal{B}_1$ (fixed allocation) does not require bit-allocation processing. This will be particularly valuable when these ideas extend to frequency-selective channels.

As mentioned before, on the other hand, choice of $\mathcal{B}_1$ and $\mathcal{B}_2$ depends on channel distribution. Thus, we assume that channel statistics as well as channel information are known to transmitter. If $\mathcal{B}_1$ and $\mathcal{B}_2$ do not match current fading statistics, it could cause significant performance loss. In Section V, we will deal with this mismatch problem.

IV. Numerical Results

We consider Rayleigh fading for $M \in \{2, 4, 6\}$ antennas. Suppose that each allocation is restricted to discrete values, $r_i = \beta B_i$, with $\beta = 0.75$ and $B_i \in \{0, \ldots 8\}$. Fig. 3 illustrates restricted-search sets, $\mathcal{B}_1$ and $\mathcal{B}_2$, optimized for Rayleigh fading, for some $B = \sum_i B_i = R / \beta$. In Fig. 3, for example, an allocation denoted as [4 2] means $B_1 = 4$ and $B_2 = 2$. Notice that, in case of $\mathcal{B}_1$, no bit is assigned to last subchannels (s_M). This is because s_M is exponentially distributed in Rayleigh fading [6], which means, information-theoretically, that it takes infinite average power to convey a nonzero rate, however small it is, over this subchannel. Especially for $M = 2$, all information is forcefully conveyed over the first subchannel, which explains why optimal allocation is not necessarily most frequently-used allocation, and which inevitably results in a significant performance loss.

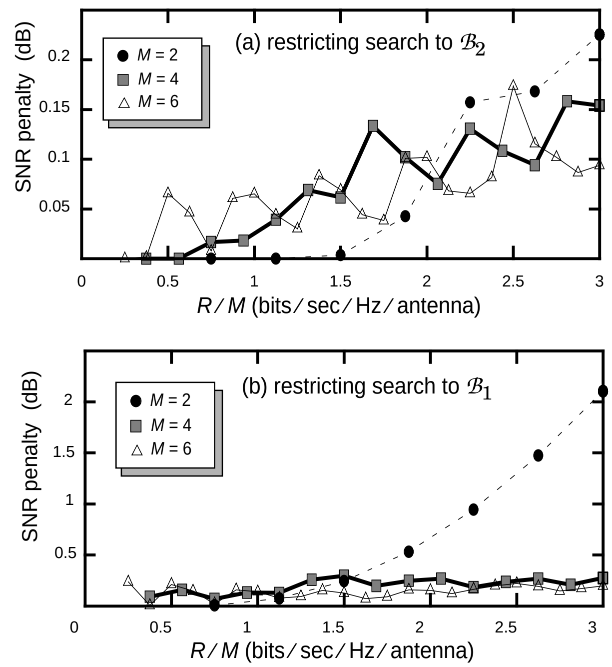

Fig. 4. Relative performance of restricted search over $\mathcal{B}_1$ in (a) and over $\mathcal{B}_2$ in (b) compared to full-search strategy in Rayleigh-fading with $M \in \{2, 4, 6\}$.

We evaluates performance of restricted search over $\mathcal{B}_1$ and $\mathcal{B}_2$ in Fig. 3, which plots average SNR penalty by using $\mathcal{B}_2$ in Fig. 4-a, and by using $\mathcal{B}_1$ in Fig. 4-b against rate per signaling interval and per antenna when compared to full-search strategy ($\mathcal{B}$). In both cases, the restricted searches perform only marginally worse. For example, restricting the search to $\mathcal{B}_2$ incurs an SNR penalty of less than 0.23 dB when $M = 2$. Even the fixed allocation ($\mathcal{B}_1$) performs well, falling only 0.3 dB short of the full-search performance for both $M = 4$ and $M = 6$. One exception is for $M = 2$, where the penalty by using $\mathcal{B}_1$ can be as large as 2.2 dB at $R / M = 3$.

V. Robust Bit-Allocation Strategy

As discussed at the end of Section III, bit-allocation based on restricted search over $\mathcal{B}_k$ could cause a mismatch problem when actual fading statistics are different from those to which $\mathcal{B}_k$ is optimized. When size of $\mathcal{B}_k$ is small, this mismatch problem can be serious. In this paper we consider mild mismatch, which occurs when statistics estimation differs from real channels or when statistics slightly change between estimations. Also we are only concerned with more than two antennas ($M > 2$).

An intuitive way to make bit-allocation robust to fading change is to increase the size of $\mathcal{B}_k$ and to select its members appropriately. Since restricting search to $\mathcal{B}_1$ is near-optimal for $M > 2$, an union of several $\mathcal{B}_1$'s, whose members are optimized to some typical fading statistics, would perform reliably. Expanding $\mathcal{B}_k$ obviously leads to increase in complexity. For mild change of statistics, however, only a few additions are sufficient.

For example, consider Ricean fading of $K = 4.45$ in $M = 4$ antenna arrays with $\beta = 0.75$ and $B_i \in \{0, \ldots 8\}$, where K denotes the Rice factor [7]. The dotted line in Fig. 5 represents SNR penalty by using a mismatched $\mathcal{B}_2$, which is optimized to Rayleigh ($K = 0$), compared to a full search over $\mathcal{B}$. This mismatch loss can be large, as illustrated in Fig. 5, more than 1 dB. By adding three $\mathcal{B}_1$'s that are optimized to $K = 0$, $K = 2.41$, and $K = 6.46$, respectively, we constitute a robust search set:

$$\mathcal{B}_{3,\mathrm{rob}} = \mathcal{B}_{1,K=0} \cup \mathcal{B}_{1,K=2.41} \cup \mathcal{B}_{1,K=6.46}\,, \qquad (5)$$

where $\mathcal{B}_{1,K=6.46}$ means that $\mathcal{B}_1$ is optimized to $K = 6.46$. Fig. 5 shows performance of robust bit-allocation strategy, where thick line corresponds to $\mathcal{B}_{3,\mathrm{rob}}$. As references, SNR penalty of $\mathcal{B}_1$ (square) and $\mathcal{B}_2$ (circle), which are optimized to actual fading ($K = 2.41$), is plotted. Clearly, bit-allocation over $\mathcal{B}_{3,\mathrm{rob}}$ performs very well, whose SNR penalty is less than 0.2 dB.

VI. Conclusions

Based on properties of MIMO fading channels, we proposed bit-allocation restricting search to restricted search sets, $\mathcal{B}_2$ (containing two allocations) and $\mathcal{B}_1$ (fixed allocation). These bit-loading strategy considerably reduce complexity while perform only marginally worse than optimal bit-allocation. For example, in Rayleigh fading $M \in \{4, 6\}$, its average SNR penalty is below 0.15 dB and below 0.3 dB when restricting search to $\mathcal{B}_2$ and $\mathcal{B}_1$, respectively. For $M = 2$, it has been found that at least two allocations must be considered ($\mathcal{B}_2$). We also proposed robust bit-allocation which can handle some variations of fading statistics.

Proposed bit-allocation strategies extend to orthogonal frequency division multiplexing (OFDM) in frequency-selective channels. If each OFDM tone is restricted to have the same bit-budget, called flat-frequency, a great deal of complexity can be saved by proposed strategies. Since frequency correlation between tones is ignored, it would incur performance loss. We will report, in the future, how flat-frequency strategy performs compared to conventional bit-allocation algorithms. In the meanwhile, we only consider a fixed total rate system in this paper. We will investigate joint optimization of bit-allocation and rate regions when the total rate is variable.

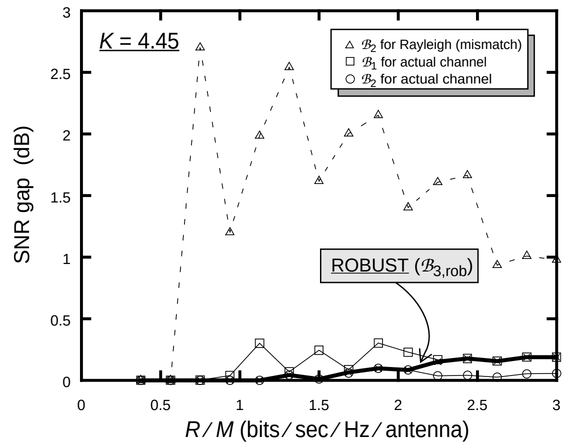

Fig. 5. Fading mismatch of restricted search over $\mathcal{B}_2$ optimized to Rayleigh fading and performance of robust bit-allocation in Ricean fading ($K = 4.45$) with $M = 4$ antennas.

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