ECE 6605
Information Theory

HW #5: Assigned October 17 2003, due October 27,2003.

1) Lempel Ziv problem This problem concerns the Lempel-Ziv (LZ) data compression algorithm. Answer parts a) and b) below. *The sources and codes in parts a) and b) are unrelated.*

a) **Encoding.** Given that the LZ encoder starts with the following table, construct the table and indicate the transmitted codewords for the source output $\underline{X}$. You may assume that the source outputs either □ or □. (You can number the codewords 1,2,3...)

X	C(X)
□	1
□	2

Source output: $\underline{X}$= □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □

b) **Decoder.** In this part you will be the decoder. Given the following sequence of codewords, decode it, and construct the table as you go along. You will start with the same table the encoder starts with and assume the source outputs two letters c and d.

X	C(X)
c	1
d	2

Codeword sequence to be decoded: 1 2 1 3 4 5 1 7 3 (decode this message and indicate whether this is a valid **encoding**. If it is not, give the correct codeword for the source output).

2) Arithmetic coding problem

Consider a binary source that outputs a's and b's independently with probabilities

P(a) = 0.1 and P(b) = 0.9

Using arithmetic coding, find the codeword for the following source output and compare it with the source entropy:

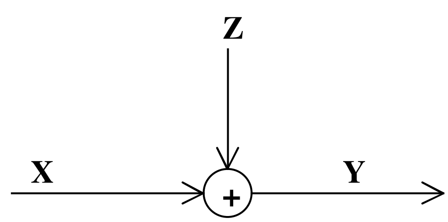
a b a b b b a a

2) Capacity of the Z-channel – (sometimes this channel is a model for a single-photon optica communication channel). The Z-channel has binary input and output alphabets and transition probabilities p(y|x) given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

4) Find the channel capacity of the following discrete memoryless channel:



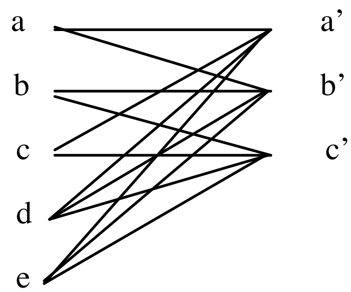
where Pr [Z = 0] =0.5 and Pr [Z = a]=0.5 and '+' is standard (real) addition. The alphabet for X is {0,1}. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

5) Channels with memory have higher capacity

Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$ where $\oplus$ is XOR mod 2 addition, and $X_i, Z_i \in \{0,1\}$. Suppose that the sequence $Z_1, Z_2, ..., Z_n$, has constant marginal probabilities $P\{Z_i = 1\} = p = 1 - P\{Z_i = 0\}$, but that $Z_1, Z_2, ..., Z_n$, are not necessarily independent. Let $C = 1 - H(p, 1-p)$, Show that the maximum mutual information per channel use exceeds C namely, show that

$$\max_{p(X_1, X_2, ..., X_n)} \frac{1}{n} I(X_1, ..., X_n; Y_1, ..., Y_n) \geq C$$

6) Suppose, we have a discrete memoryless channel (DMC) with 5 inputs and 3 outputs, say {a,b,c}. Suppose, this channel is graphically represented as follows:



where the probability of transitions from a, b, and c to a', b', and c' are all 1/2. However, the probability of transitions from d to a', b', and c' are all 1/3, and probability of transitions from e to a', b', and c' are 1/2, 1/4, and 1/4 respectively.

a. Let W denote the channel transition matrix. Write down this matrix, and provide a two dimensional picture to display the convex hull of its columns.
b. Can you suggest a DMC with a fewer number of inputs which can be used instead of above channel for all input distributions?
c. In class we have seen that for a given input distribution, it is possible to find a distribution over just three of the input symbols that yield the output distribution. Suppose that for a given input distribution, none of the output symbols have probability of 1/2. Then, what are the symbols that we must necessarily keep them to obtain the same output distribution?
d. Suppose, we omit symbols d and e from the channel to obtain a 3 by 3 channel. Describe the channel transition matrix W_1? If we concatenate two copies of this channel, to get W_2, plot the convex hull of the columns of W_2 against the convex-hull of columns of W_1. How do you compare the mutual information of these channels for any input distribution?
e. Now, suppose a concatenation of W_1 n times, to obtain W_n. Plot the convex hull of the columns of W_n against the convex-hull of columns of W_{n-1}. How does the mutual information changing as n increases? What is the limit of mutual information as n goes to infinity?