Numerical Dispersion of Higher Order Nodal Elements in the Finite-Element Method

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***Abstract*— The discretization inherent in the finite-element method results in numerical dispersion. In this work, this dispersion is investigated for a time-harmonic plane wave propagating through an infinite, two-dimensional, finite-element mesh composed of uniform quadrilateral and triangular elements. The effects on the dispersion due to the propagation direction of the wave, the order of the elements, the node density, and the mesh geometry are studied. Results are given which can serve as a guide in selecting the appropriate element order, node density, and mesh geometry when applying the finite-element method.**

I. Introduction

The FINITE-ELEMENT method is used extensively to solve field problems. To have confidence in the technique, it is important to understand the errors associated with the method. One of the most significant errors arises from the inability of the polynomial basis functions to represent the fields exactly within an element. A wave propagating through a finite-element mesh will experience numerical dispersion as a result of this error.

The numerical dispersion associated with the use of first-order elements in a two-dimensional mesh has been investigated [1], [2]. However, the numerical dispersion associated with the use of higher order elements in a two-dimensional mesh has not been thoroughly investigated. To the authors' knowledge, the only consideration of two-dimensional, higher order nodal elements has been restricted to plane waves propagating along the axis of a mesh of higher order quadrilateral elements. With this restriction, the results for the two-dimensional elements reduce to those for the one-dimensional elements. Second-order one-dimensional elements were investigated in [3], and second- through eighth-order one-dimensional elements were investigated in [4].

In many applications it is necessary to utilize higher order elements to achieve the required precision with limited computational resources. In these situations, it is desirable to know the numerical dispersion associated with these elements. As a result, the numerical dispersion of a time-harmonic plane wave propagating through an infinite, uniform, two-dimensional, finite-element mesh composed of higher order, quadrilateral, and triangular nodal elements is investigated in this work. This dispersion can be characterized by a cumulative phase error. It is shown here that the phase error is a function of node density, direction of propagation of the plane wave, order of the elements, and orientation of the elements within the mesh. This information can be used as a guide in determining the proper element order, node density, and mesh geometry when applying the finite-element method.

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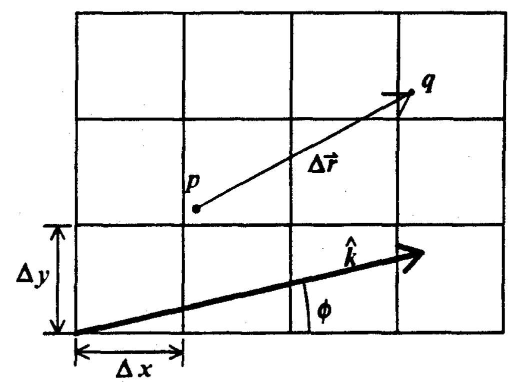

Fig. 1. Diagram depicting a plane wave propagating through a portion of an infinite, uniform, finite-element mesh.

II. Dispersion Analysis

Consider an infinite, linear, two-dimensional, homogeneous, isotropic, and source-free region. The field in this region is governed by the Helmholtz equation

$$\nabla \cdot (\nabla E) + k^2 E = 0 \tag{1}$$

where an $e^{j\omega t}$ time dependence is assumed and $k = \omega\sqrt{\mu\epsilon}$ is the analytical wavenumber.

Now consider dividing the region into an infinite uniform mesh of finite elements. The field in this mesh will be governed by a discretized Helmholtz equation. One possible field that can exist is a plane wave propagating at an angle ϕ. Since the plane wave within the mesh is approximated in terms of expansion functions, it will propagate with a numerical wavenumber $\tilde{k}$ that differs from the analytical wavenumber k. While the relationship between the field at any two points in the analytical case is a simple phase factor, the relationship between the field at any two points in the finite-element case is, in general, more complicated. However, if q is any point within an element and p is any other point at the same relative position within another identical element (as in Fig. 1), then the relationship between the discretized field at these two points $\tilde{E}_q$ and $\tilde{E}_p$ is

$$\tilde{E}_q = \tilde{E}_p e^{-j\tilde{k}\hat{k}\cdot\Delta\vec{r}} \tag{2}$$

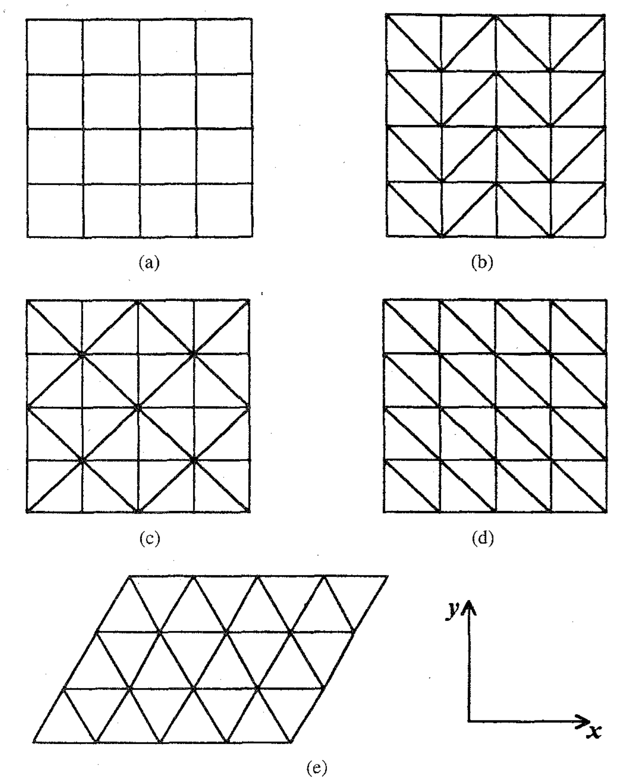

Fig. 2. Schematic representations of the uniform, infinite, two-dimensional finite-element meshes: (a) quadrilateral mesh, (b) arrow mesh, (c) diamond mesh, (d) one-directional mesh, and (e) hexagonal mesh.

where $\Delta\vec{r}$ is the vector from the point p to point q and $\hat{k} = \cos\phi \hat{a}_x + \sin\phi \hat{a}_y$.

The goal in the dispersion analysis is to determine the relationship between the numerical wavenumber $\tilde{k}$ and the analytical wavenumber k. This is accomplished by using the relationship given by (2) in conjunction with residual equations for the unknowns which are obtained from the finite-element formulation [5]. A system of nonlinear equations is obtained which is solved for the numerical wavenumber when the analytical wavenumber is known. To facilitate the dispersion analysis process for complicated meshes, a computer program was written to generate and solve the nonlinear equations.

III. Results

The various meshes investigated in this work are shown in Fig. 2. For any of the meshes, when composed of nth-order elements the shortest distance between nodes is h and the length of the shortest side of any element is nh. The names for the meshes are given in Fig. 2. The numerical wavenumber $\tilde{k}$ for each type of mesh when composed of first-, second-, and third-order elements was calculated using the process outlined previously. The resulting phase error, in degrees per wavelength, is defined as

$$\delta_p = 360\left|\frac{\tilde{k} - k}{k}\right|. \tag{3}$$

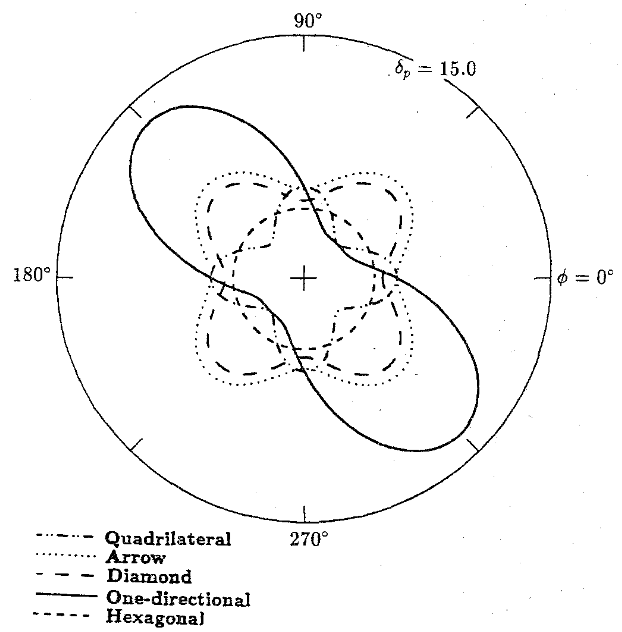

Fig. 3. Polar graph of the phase error in degrees per wavelength for the different first-order, finite-element meshes as a function of ϕ for $\lambda/h = 10$.

Figs. 3–5 are polar graphs of the phase error as a function of the angle of propagation ϕ for first-, second-, and third-order elements, respectively. In these plots, $\lambda/h = 10$. Note the scale change on the graphs; the value of the phase error at the outer edge is indicated on each graph. The plots of the phase error for larger values of λ/h possess the same shape but have a smaller magnitude. As can be seen, the phase error for each mesh is anisotropic; it varies with the direction of propagation. In addition, the severity of the anisotropic behavior varies with the mesh choice.

It is interesting to examine the relative behavior of the one-directional, arrow, and diamond meshes. All are constructed of elements that are identical right-angle isosceles triangles. From the graphs, however, it is evident that the different arrangement of the triangular elements within the mesh strongly affects the behavior of the phase error. The one-directional mesh is very asymmetric and results in a very asymmetric phase error. The arrow and diamond meshes are less asymmetric and result in less asymmetric errors.

To estimate the error in a practical application, one often uses a worst-case scenario. Since the angular dependences of the error for the various meshes differ, it is difficult to use Figs. 3–5 to get the worst-case error. Therefore, a worst-case comparison is presented in Fig. 6, where the maximum phase error with respect to ϕ is graphed as a function of the node density for each mesh. As can be seen, the error becomes smaller as the electrical size of the element is decreased. For nth-order elements, the error is seen to be proportional to $(h/\lambda)^{2n}$ for large values of λ/h, thus, $\tilde{k}$ converges at the rate $O[(h/\lambda)^{2n}]$. This is a superconvergent rate rather than an ordinary rate of $O[(h/\lambda)^{n+1}]$ [6]. The rate of convergence is not a function of propagation direction or mesh type.

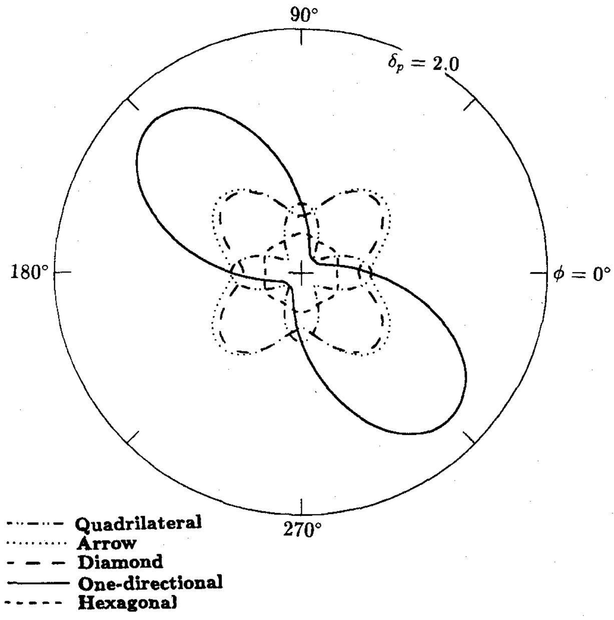

Fig. 4. Polar graph of the phase error in degrees per wavelength for the different second-order, finite-element meshes as a function of ϕ for $\lambda/h = 10$.

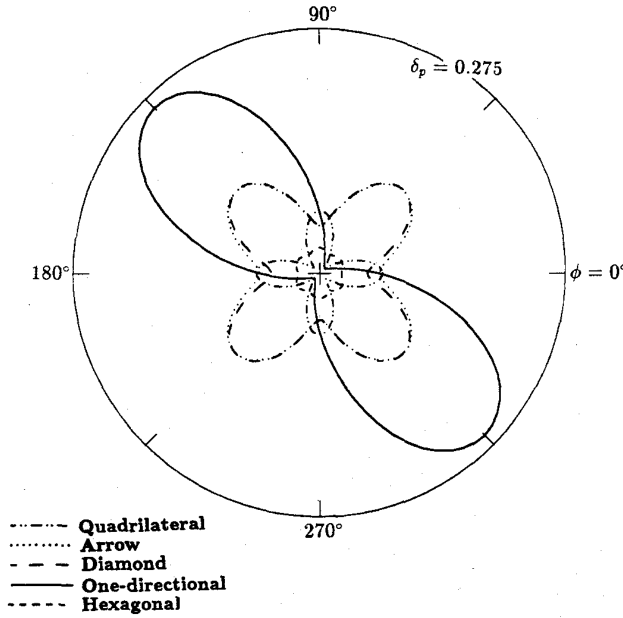

Fig. 5. Polar graph of the phase error in degrees per wavelength for the different third-order, finite-element meshes as a function of ϕ for $\lambda/h = 10$.

Fig. 6 indicates that the higher order elements dramatically decrease the phase error. For instance, to maintain a maximum phase error of 0.1 degrees per wavelength for the quadrilateral mesh requires a value of $\lambda/h = 75$ if first-order elements are used, $\lambda/h = 16$ if second-order elements are used, and $\lambda/h = 9$ if third-order elements are used. Thus, when the maximum phase error is 0.1 degrees per wavelength, the number of unknowns per square wavelength is approximately $(75/16)^2 \approx 22$ times greater for the mesh of first-order elements compared with second-order elements and is approximately $(75/9)^2 \approx 70$ times greater for the mesh of first-order elements compared with third-order elements.

It is interesting to compare the phase error of the first- and second-order nodal elements to the phase error of the first- and second-order tangential-vector elements [5], [7], [8]. The phase error for the quadrilateral nodal elements is identical to that of the quadrilateral tangential-vector elements. However, the phase error for the triangular nodal elements is different than that of the triangular tangential-vector elements [5], [8]. Note that the approximations of the phase error for the quadrilateral tangential-vector elements developed in [7] can also be used as approximatioins for the phase error of the quadrilateral nodal elements.

The investigation of the phase error in this communication has been restricted to node densities, λ/h, greater than eight nodes per wavelength. The finite-element method is not normally used at node densities less than this. However, when lower node densities are used, the solution of the dispersion relations for some of the meshes in this communication can result in a complex value of the numerical wavenumber, $\tilde{k}$. This means that the plane wave will experience attenuation in addition to a cumulative phase error as it propagates through the mesh. Such behavior has been reported in higher order

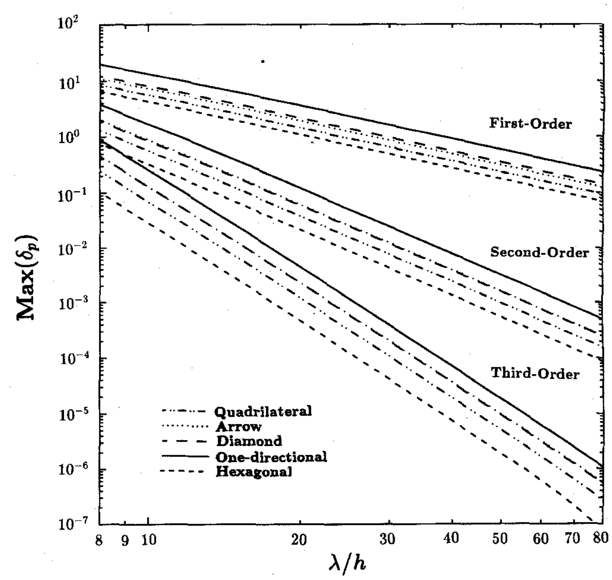

Fig. 6. Graph of the maximum phase error with respect to ϕ in degrees per wavelength for the different finite-element meshes, as a function of λ/h.

one-dimensional and first-order two-dimensional meshes [1], [4]. Based on the results reported in [4], this behavior can be expected to occur when λ/h is less than approximately $2n$, where n is the order of the element. The exact node densities at which this behavior occurs will vary with the direction of propagation of the plane wave through the mesh and the type of mesh.

IV. Conclusion

The phase error that results from the numerical dispersion inherent in the finite-element method when using nodal elements has been investigated. This error is a function of element order, node density, and mesh type, and it decreases with increasing element order. As a result, substantial computational savings can be realized by using the higher order elements when accuracy is an issue. In addition, the phase error decreases with an increase in node density; the numerical wavenumber $\tilde{k}$ for a n^{th}-order nodal finite element is seen to converge at a superconvergent rate of $O[(h/\lambda)^{2n}]$.

The size and angular dependence of phase error is a function of the shape and arrangement of the elements. Three arrangements of the same element are shown to have significantly different phase errors. Care has to be exercised in choosing the arrangement, as well as the shape of the elements. Of the meshes considered, the hexagonal mesh is the best; it has the lowest maximum phase error and the least dependence on the angle of propagation. The one-directional mesh is the worst; it has the largest maximum phase error and the most dependence on the angle of propagation.

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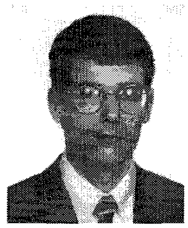

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