A Study of the Applications of Analog Computers

V. P. KODALI, MEMBER, IEEE

Abstract—**An electronic analog computer is a tool for solving mathematical differential equations. The basis for the use of such a computer in the simulation of many scientific and engineering problems is the possibility of characterizing these situations as formal mathematical systems by using differential equations. In this paper, following a brief description of the computing elements, the applications of analog computers in the study of various linear, nonlinear, and time-varying systems are described with typical illustrative examples from diverse fields of interest. Computer applications in solving partial differential equations are also discussed. Potential capabilities and limitations of this type of computer as an aid in research and analysis are given.**

Introduction

Any two systems are said to be the analogs of each other if their activity permits itself to be described by similar mathematical formalism or set of equations. Accordingly, the study of analogs is, to a large extent, a mathematical study. It is apparent that different devices or systems could be mutually analogous without necessarily bearing any physical resemblance. In fact, the analogs of many hydraulic, pneumatic, mechanical, and electrical or electronic phenomena exist in other disciplines of study. A variety of problems associated with these and other physical systems lend themselves to the use of analog computers in their analysis, simulation, and design. A treatment of the principles of operation of analog computers is therefore a matter of interest to many.

Study of physical systems using the electronic analog computers is broadly understood as involving two stages. In the first stage, the problem is characterized or mathematically described by using differential equations. Simpler situations may be described through a single equation. But more complex systems, in general, lead to a set or sets of simultaneous differential equations. The second stage consists in translating these mathematical equations into computer circuits and transferring them on to the analog computer. An analog computer is best suited for solving systems of ordinary differential equations, linear or nonlinear, with constant or time-varying coefficients. Analog computers have also been used, with some success in the past, to solve problems involving partial differential equations.

This paper is a theoretical study and provides an introductory account about analog computers and their application to the study of engineering systems and physical phenomena. One can hardly hope to gain a complete understanding of the role of this class of computers in the study of systems from a brief summary. Consequently, no attempt is made at completeness. Instead, some rather simple but typical problems are considered as examples to present a unified account of the use of analog computers. Here, the approach will be to group various systems into three broad categories. The classification is based on the type of the differential equation, which characterizes the system. Under each class, a variety of physical systems will be considered and their representation on the analog computer deduced. A brief description of various analog computer components precedes this study. The paper concludes with a discussion of some general limitations to the use of analog computers.

Manuscript received March 30, 1966; revised September 28, 1966.
The author is with the Department of Electrical and Electronic Engineering, University of Leeds, Leeds, England.

The Computing Elements

The principal component in an analog computer is the operational amplifier [1]–[6], which is essentially a direct coupled amplifier with very high open-loop gain. Open-loop dc-to-dc gains of the order of 100 000 are typical in better equipment. When two impedances Z_i and Z_f are connected to such an amplifier as shown in Fig. 1(a), it can be shown [7] that the input and output voltages to the circuit are related by

$$e_o = -\left[\frac{Z_f}{Z_i}\right] e_i. \tag{1}$$

Thus, the quantity Z_f/Z_i acts as an operator on the input voltage e_i to produce the output e_o. When Z_f and Z_i are chosen to be two resistors, the operator simply becomes a multiplying constant and (1) takes the form

$$e_o = -\left(\frac{R_f}{R_i}\right) e_i. \tag{1a}$$

This type of circuit is known as the inverting amplifier because of the phase inverting property (negative sign) associated with it. Most inverting amplifiers, which are part of the standard computing setup, usually have multiplication constants of 1, 4, and 10. Other values may be realized by using potentiometers in conjunction with the amplifiers. The governing equation for a potentiometer is given by

$$e_o = pe_i, \tag{1b}$$

where the quantity p is continuously variable between the limits of 0 and 1.

In general, the amplifier may have more than one input port as shown in Fig. 1(d). The result is a summing amplifier, whose output e_o is given by

$$e_o = -\left[\frac{R_f}{R_1}e_1 + \frac{R_f}{R_2}e_2 + + + \frac{R_f}{R_n}e_n\right]. \tag{1c}$$

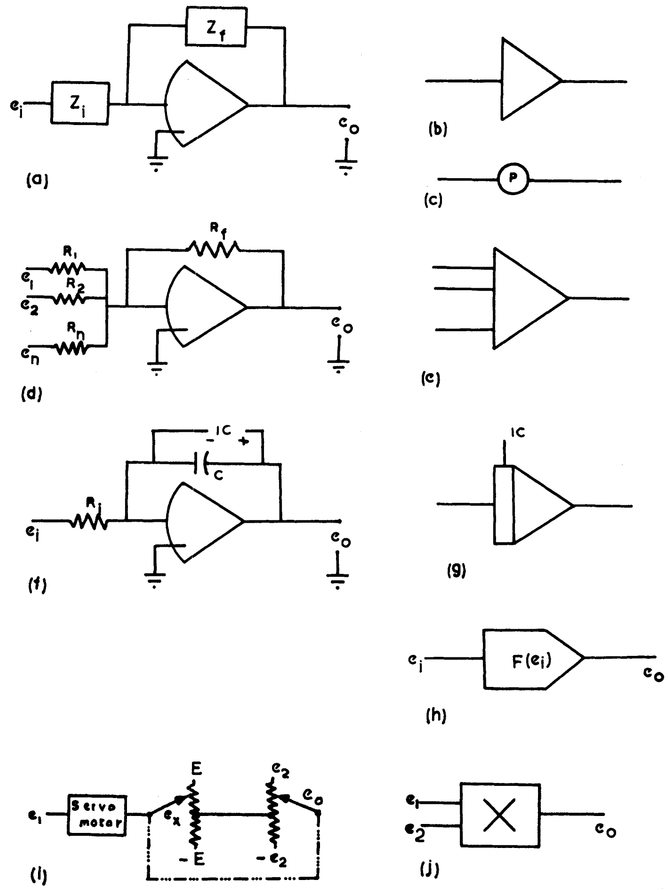

Fig. 1. Computing elements and their symbolic representation. (a) High gain dc amplifier with feedback and input impedances. (b) Symbol for multiplying-inverting amplifier. (c) Symbol for the potentiometer. (d) Summing amplifier. (e) Symbol for summing amplifier. (f) Integrating amplifier (IC denotes the initial conditions). (g) Symbol for the integrator. (h) Symbol for a function generator. (i) Schematic of a servo-multiplier (broken line indicates mechanical but no electrical connection). (j) Symbol for function multiplier.

When a capacitor of impedance $1/sC$ and a resistor of resistance R_i are connected for Z_f and Z_i, respectively, as shown in Fig. 1(f), then

$$e_o = -\frac{1}{R_iCs}e_i = -\frac{1}{R_iC}\int_0^t e_i \cdot dt + IC. \quad \text{(1d)}$$

This connection, therefore, enables the use of the operational amplifier as an integrating amplifier. The quantity IC in (1d) denotes the constants of integration. This "initial condition" is introduced into the electronic circuit as the charge across the capacitor at the start of the problem.

At least in theory, it is possible to use the operational amplifier to perform mathematical differentiation by connecting a resistor in the place of Z_f and a capacitor in the place of Z_i. This is, however, avoided in actual practice because of the undesirable noise amplification property associated with such a type of circuit [2], [6].

A number of function generators and multipliers are included in the class of nonlinear computing elements. These may be electrical or mechanical devices or a combination of the two. One of the most commonly used of such devices is the servo-driven multiplier, sketched in Fig. 1(i). A typical servo multiplier consists of at least two multi-turn identical potentiometers, whose shafts are ganged and coupled to the rotor shaft of a servo motor. The shaft of the motor, which is actuated by one of the input signals e_1, rotates and sets the arm of the first potentiometer, called the "follow-up" or the "error-sensing" potentiometer, to indicate a voltage $e_x = e_1$. Since the shafts of the potentiometers are mechanically (but not electrically) coupled, the angular shift in the position of each potentiometer shaft is the same. Hence, the rotating arm of the second potentiometer, called the multiplying potentiometer, is set at a point which corresponds to an output voltage $e_o = e_1e_2/E$. Here, e_2 is the voltage applied to the multiplying potentiometer and E is the reference voltage (usually 100 volts) applied to the follow-up potentiometer. This setup thus enables the mathematical multiplication of the two quantities represented by e_1 and e_2.

Time-division multipliers and quarter-square multipliers are the other frequently used multiplying devices. The latter are known to possess better frequency response and provide higher accuracy [1], [8].

Linear Systems with Time-Invariant Elements

As an example of a simple mathematical system, the following second-order linear differential equation[1] with constant coefficients is considered.

$$m\ddot{y} + n\dot{y} + ky = F. \quad (2)$$

Equation (2) is typical of the type that is often encountered in the study of servomechanisms or in the study of many basic mechanical systems like the rectilinear system with a single rigid mass, a linear restoring spring, and viscous damper. The following two comments help in understanding the derivation of the computer circuit, which is used to simulate the system given in (2).

1) The equation is rewritten as

$$\ddot{y} = \frac{F}{m} - \frac{n}{m}\dot{y} - \frac{k}{m}y \quad \text{(2a)}$$

from which, we observe that $\ddot{y}$ may be obtained by algebraically *adding* the three quantities F/m, $-(n/m)\dot{y}$, and $-(k/m)y$.

2) $\dot{y}$ and y are computed by *integrating* $\ddot{y}$ once and twice, respectively.

The computer diagram is shown in Fig. 2.

Subject to appropriate simplifying assumptions, a large number of problems involving the study of torsional vibrations of a disk connected at the end of a shaft, the oscillations of a cantilever supporting a mass at its free end, the unbalance in a rotating machine,

[1] The notations $\dot{y}$ and $\ddot{y}$ are used to denote the first and second derivatives of y with respect to time.

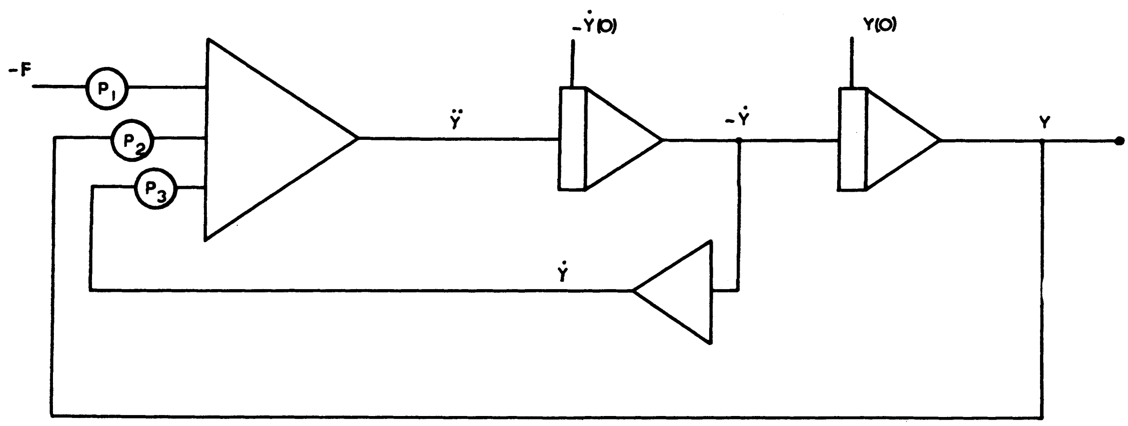

Fig. 2. Computer circuit for solving the equation $m\ddot{y}+n\dot{y}+ky=F$. Potentiometer settings: $P_1=1/m$; $P_2=k/m$; $P_3=n/m$.

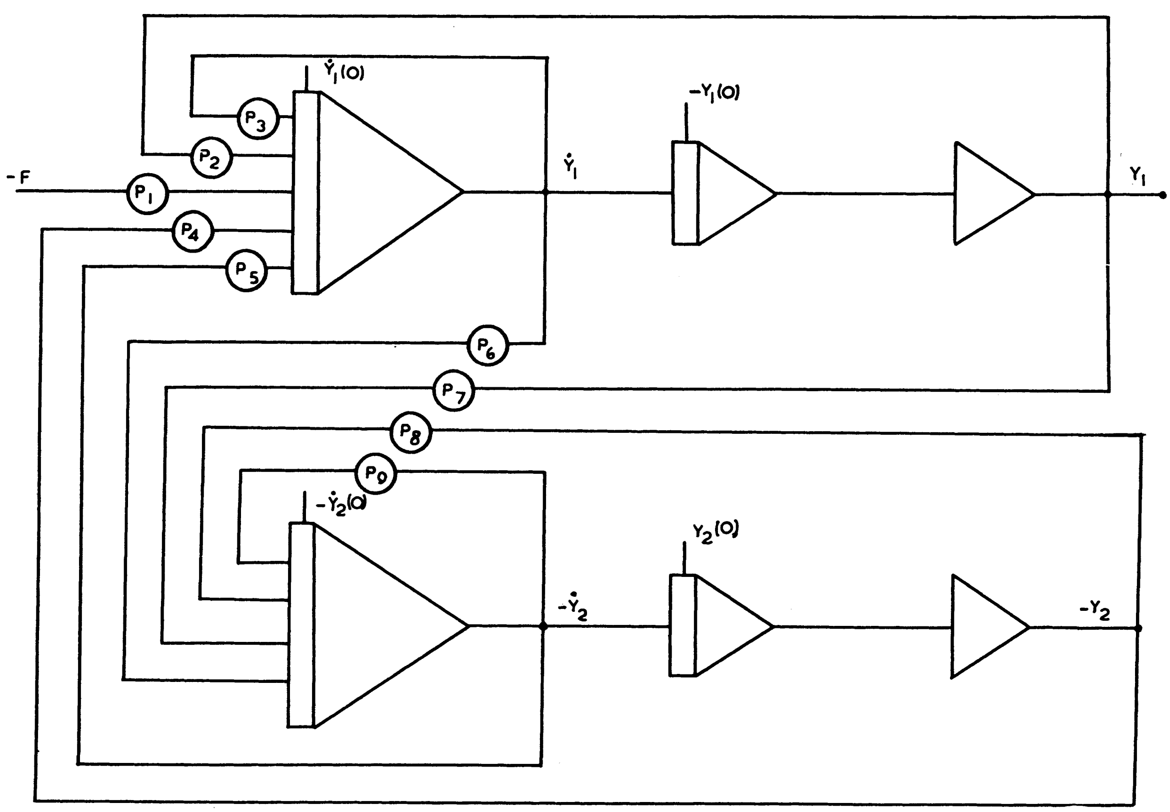

Fig. 3. Computer circuit for solving the simultaneous differential equations given in (3). Potentiometer settings: $P_1=1/m_1$; $P_2=k_1/m_1$; $P_3=n_1/m_1$; $P_4=b_2/m_1$; $P_5=a_2/m_1$; $P_6=a_1/m_2$; $P_7=b_1/m_2$; $P_8=k_2/m_2$; $P_9=n_2/m_2$.

and the disturbance suffered by a spring-loaded flyball governor rotating with a constant angular velocity reduce to the mathematical system given in (2) [9]–[11].

Analog computers can also be used to solve simultaneous differential equations. Figure 3 gives a computer circuit for solving the following set of simultaneous differential equations.

$$m_1\ddot{y}_1 + n_1\dot{y}_1 - a_2\dot{y}_2 + k_1y_1 - b_2y_2 = F$$
$$m_2\ddot{y}_2 + n_2\dot{y}_2 - a_1\dot{y}_1 + k_2y_2 - b_1y_1 = 0. \qquad (3)$$

Here two circuits are set up, each solving one differential equation, and then interconnections between the two circuits are made as required.

In practice, a set of simultaneous differential equations represent systems with more than one degree of freedom. The oscillatory motion of a dynamic absorber with viscous damping or similar mechanically coupled systems, a number of thermal systems involving heat transfer, many problems in automatic control and flight simulation, and an amplidyne system consisting of rotating machinery are some examples of the physical systems that are mathematically described by sets of simultaneous differential equations [9], [12]. The number of equations in each case depends on the number of degrees of freedom. A derivation of the corresponding computer diagram, simply, consists in drawing one cir-

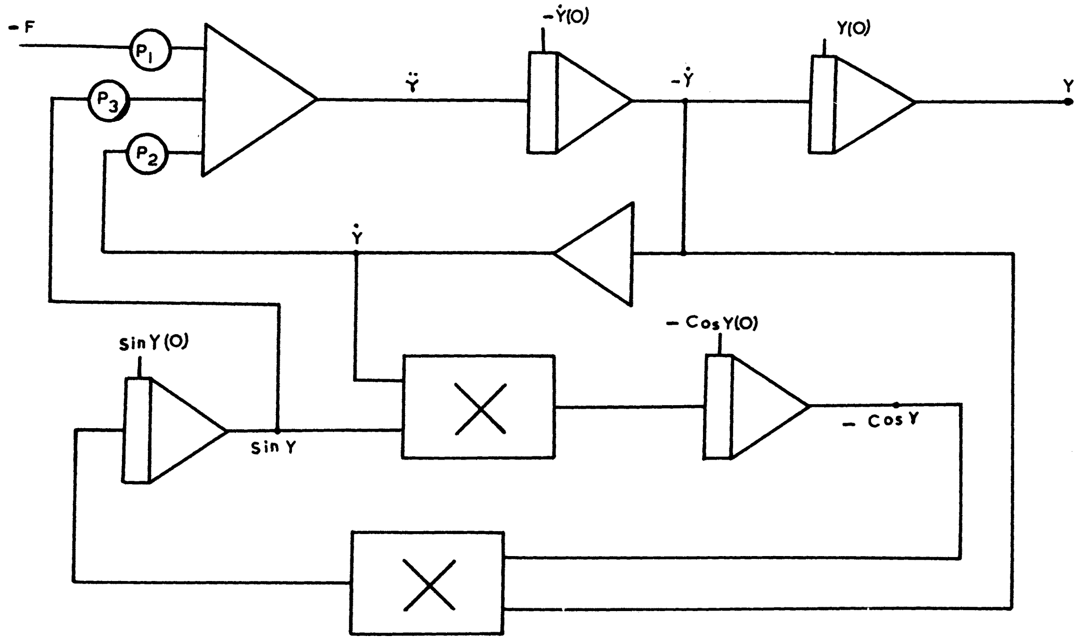

Fig. 4. Computer circuit for solving the nonlinear differential equation $\ddot{y}+c\dot{y}+k(\sin y)=F$. Potentiometer settings: $P_1=1.0$; $P_2=c$; $P_3=k$.

cuit for each equation and then making the necessary interconnections.

Nonlinear Systems

The situations discussed so far have the mathematical representation described by ordinary linear differential equations with constant coefficients. Normally, such equations represent ideal systems. In reality, however, many physical systems possess some form of nonlinearity or periodic fluctuation [13]–[15]. Consequently, many of the examples cited in the above section will, in fact, need to be considered as nonlinear systems unless a number of simplifying assumptions are made. Some of the more common forms of nonlinearities include saturation, dead-zone, coulomb friction, and hysteresis. A variety of special electronic circuits useful in simulating these and other types of nonlinear phenomena are described in the literature [6], [12], [16]. These function generators make use of relays, diode circuits, etc., in conjunction with the computing elements listed in Fig. 1.

Nonlinear springs and viscous dampers are inherent in many mechanical systems, a typical example being a spring-damper assembly with a nonlinear restoring spring. The vibrations in such an assembly due to an external disturbing force $F(t)$ are described by Duffing's equation

$$m\ddot{y} + c\dot{y} + k(y)y = F(t), \tag{4}$$

where m, c, and $k(y)$ are the mass, viscous friction, and the spring constant of the mechanical assembly, respectively. Here $k(y)$ is a nonlinear function of the displacement y.

Figure 4 gives a computer circuit for solving the nonlinear differential equation

$$\ddot{y} + c\dot{y} + k(\sin y) = F. \tag{5}$$

This equation describes the oscillations of a simple pendulum suspended by a massless, inelastic thread when its amplitude of oscillation is not restricted to be small. An analogous equation arises in studies concerning the stability of synchronous motors [17], [7].

Van der Pol's equation describing the forced oscillations in an electronic circuit containing a nonlinear resistor, Lane-Emden's equation dealing with the gravitational equilibrium of a gaseous configuration (star) in a stellar structure, and the nonlinearity arising from iron cored apparatus in electrical power circuits are some of the well-known nonlinear phenomena in diverse fields of study [15]. A detailed mathematical treatment of several nonlinearities which occur in engineering and physical sciences appears in the literature cited [13]–[15].

Time-Varying Systems

A class of physical systems, which are described by linear or nonlinear differential equations with time-varying coefficients, are classified as the time-varying systems. Time-varying coefficients denote the time-varying nature of the parameters or component values, the variation being caused by an agent which may be either external or internal to the system. Aerodynamical action is a familiar example of an external agent, while the unbalance in a rotating machine is internal to the system.

The most general form of a second-order linear differential equation with time-varying coefficients is Riccati's equation, written in the form

$$\ddot{r} + f(t)\dot{r} + ag(t)r = 0. \tag{6}$$

The oscillations of a simple pendulum suspended from

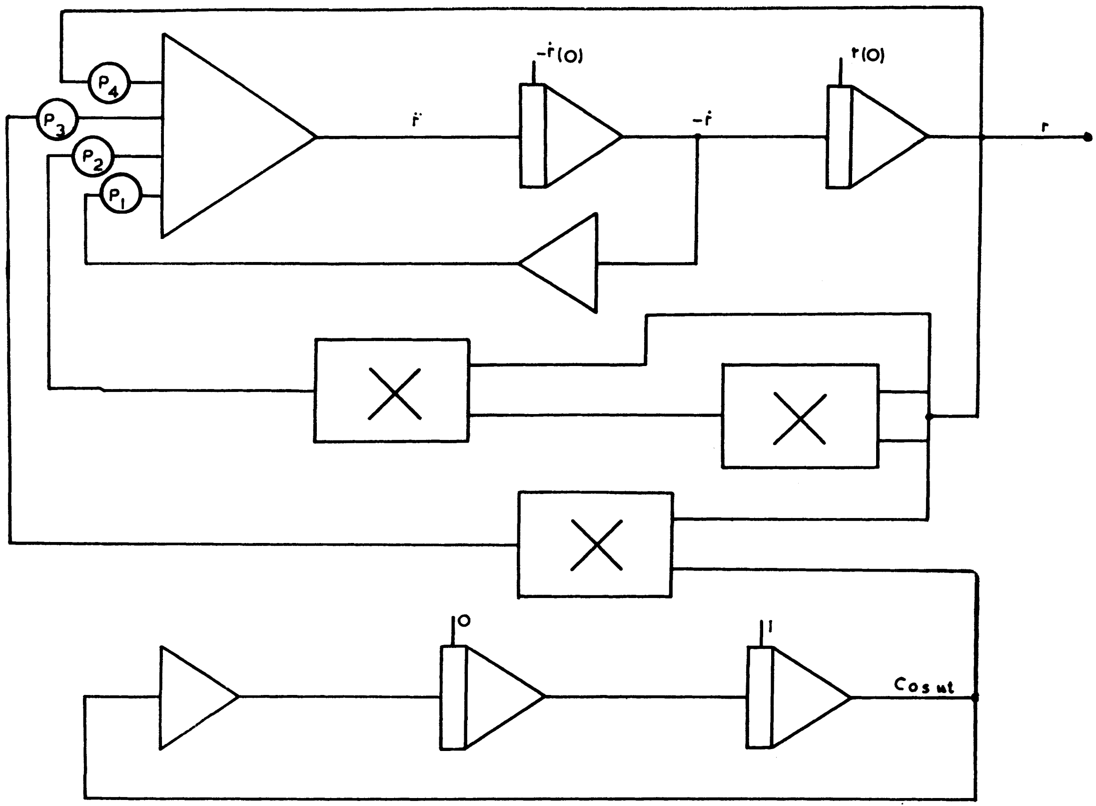

Fig. 5. Computer circuit for simulating Melde's system. Potentiometer settings: $P_1=c$; $P_2=b$; $P_3=1.0$; $P_4=a$.

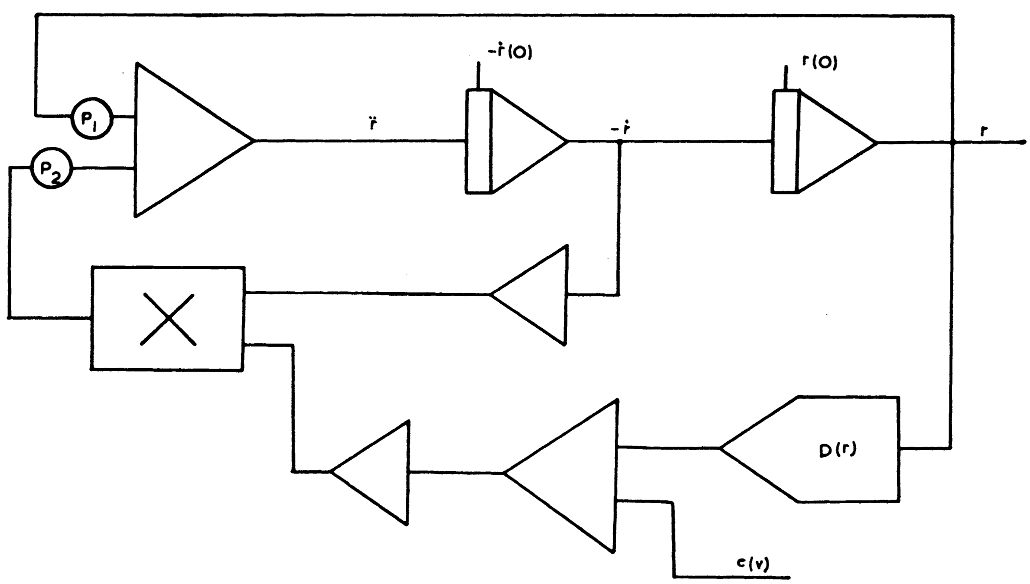

Fig. 6. Computer circuit for studying the oscillations of a suspension bridge. Potentiometer settings: $P_1=a$; $P_2=1.0$.

a vibrating support constitute a Riccati system. Mathieu's equation characterizing a variety of periodic phenomena in mechanical and electrical circuits, Gauss's equation stating some basic results in noise theory, and equations describing the combustion of gases in flames are some of the time-varying systems [15].

A computer circuit is given in Fig. 5 for solving the following nonlinear differential equation with a periodic coefficient.

$$\ddot{r} + c\dot{r} + (a + br^2 + \cos \omega t)r = 0. \tag{7}$$

This equation [18] describes the oscillations in Melde's experiment, in which one end of a horizontal thread is fixed while the other end is attached to the prong of a massive low-frequency tuning fork. An analogous situation is the vibratory motion of a long uniform column subjected to a constant axial load with a superposed sinusoidal ripple force due to an unbalance in the rotating machinery. The vibrational characteristics of a loudspeaker diaphragm in relation to subharmonics are similar, in certain respects, to those described by (7).

The oscillations of a suspension bridge due to aerodynamical action of wind [19] are represented by the equation

$$\ddot{r} + \{c(v) + D(r)\}\dot{r} + ar = 0, \tag{8}$$

where $c(v)$ denotes the aerodynamical damping which is a function of the wind velocity. $D(r)$ is the damping coefficient for the mechanical structure. When the aerodynamical damping is in phase-opposition to the structural damping, the amplitude of self-excited oscillations grows while $|c(v)| > D(r)$. In fact, when the wind velocity exceeds a certain critical value, the aerodynamical damping is always negative. To avoid the catastrophic consequences, the designer must assure that $|c(v)| < D(r)$ at all values of the expected wind velocity. Figure 6 enables the simulation of this phenomena on an analog computer.

Applications in Solving Partial Differential Equations

Partial differential equations arise in studies related to the phenomenological laws of electromagnetic wave propagation, acoustics, fluid dynamics, heat transfer, plasticity, and the quantum theory of matter [1], [15], [20], [21]. A generally successful approach to the solution of partial differential equations, using the analog computer, involves the use of finite difference approximations. With the aid of these approximations, it is possible to represent a many-dimensional problem as a set of coupled ordinary differential equations through repeated differencing. The accuracy of the solution is improved when a smaller interval is chosen, i.e., when more interpolation stations are used.

In order to introduce the concept of difference approximations, a function $\zeta(x, t)$ is considered in two variables x and t. For the sake of continued development, it is assumed that the range of the function $\zeta(x, t)$ may be divided into N equal intervals of width δ each. An application of the known results from numerical analysis [22] now yields the following approximations:

$$\left.\frac{\partial\zeta}{\partial x}\right|_{x=n-\frac{1}{2}} \doteq \frac{\zeta_n - \zeta_{n-1}}{\delta}$$
$$\left.\frac{\partial^2\zeta}{\partial x^2}\right|_{x=n} \doteq \frac{\zeta_{n+1} - 2\zeta_n + \zeta_{n-1}}{\delta^2}$$
$$\left.\frac{\partial^3\zeta}{\partial x^3}\right|_{x=n-\frac{1}{2}} \doteq \frac{\zeta_{n+1} - 3\zeta_n + 3\zeta_{n-1} - \zeta_{n-2}}{\delta^3}, \tag{9}$$

where $\zeta_0, \zeta_1, \cdots, \zeta_n, \cdots, \zeta_N$ are the values of the functions $\zeta(x, t)$ at points $x_0, x_1, \cdots, x_n, \cdots, x_N$ in the x-space.

Equation (9) permits the rewriting of a given partial differential equation as a set of simultaneous ordinary differential equations. One differential equation results at each of the interpolation stations $0, 1, \cdots, n, \cdots, N$. For a particular multidimensional function, the larger the number of interpolation stations considered, the more will be the number of ordinary differential equations in the coupled set.

To serve as an illustration, the equation describing the electrostatic field in a charge-free region is considered in the form

$$\frac{\partial^2\Phi(x, t)}{\partial x^2} + \frac{\partial^2\Phi(x, t)}{\partial t^2} = 0. \tag{10}$$

By virtue of the results stated in (9), this equation may be written as

$$\frac{1}{\delta^2}(\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}) = -\frac{d^2\Phi_n}{dt^2}$$
$$\text{for } n = 0, 1, \cdots, n, \cdots, N. \tag{10a}$$

As a result, the problem at hand is transformed into one requiring the solution for $N+1$ simultaneous ordinary differential equations. A method for obtaining the solution for a set of such equations was explained in an earlier section.

The approach described above is a completely general one and permits itself to be readily extended to solve problems involving multidimensional space. One of the fields in which such problems are often met is quantum mechanics. A most familiar example from this field is the wave equation describing the particle in a three-dimensional space

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi(x, y, z, t) + f(r)\Psi(x, y, z, t) = 0. \tag{11}$$

Here, the problem is to find the eigenvalues and corresponding eigenfunctions subject to the prescribed boundary conditions. A number of attempts to use the analog computer for obtaining the solutions to this problem are described in the literature [23].

General Considerations Pertaining to Computer Utilization

The possibility of characterizing physical systems as mathematical systems using the differential equations provides the basis for the use of analog computers in their analysis, simulation, and design. Some of the basic principles related to such a utilization have been treated in this paper. The computer application to the study of a more complicated system normally involves the division of the system into a number of subsystems or interconnected blocks in the block-diagram notation. It is then possible to analyze each of these blocks as linear, nonlinear, or time-varying systems. In an actual computer circuit, the appropriate interconnections between these blocks result in the analog for the complete system.

During the past decade, the analog computers have found novel applications in the analysis of statistical data, modeling of various business and economic cycles, study of industrial processes and control, and in the simulation of a variety of biological and man-machine systems [24]. Simulation of nuclear reactions, problems in astronomy and trajectory tracking and flight control may be carried out on the analog computer. Advantage is often taken from an additional facility, wherein the time taken by the computer to solve the problem may be varied in direct relation to the actual time required for the phenomena to occur. This is called time scaling [1]–[4]. Thus astronomical changes, which in reality take several years, may be speeded up on the computer so that final results are observed within a few minutes or seconds. On the other hand, nuclear processes may be slowed down so as to enable the monitoring of the intermediate stages.

The treatment so far concerned the capabilities of the analog computer as a research tool. This should not, however, convey the impression that there are few limitations to this application. There are, in fact, a number of well-known limitations associated with this class of computer [1], [2], [20] and some of these will be listed in the following. The most important of the limitations orginates from the fact that the type of problems which are best suited for analog computer applications are the ones involving only a time-like variable as the independent variable. While the finite difference approximations permit the extension of the scope of applications, this is not without involving the use of relatively more equipment or proportionately more time to yield a solution. For this reason, computer application for multidimensional problem analysis has not advanced beyond the stages of academic interest. Secondly, the problem setting on an analog computer involves magnitude scaling—the process through which a linear relationship is established between the voltage at any reference node and the variable represented by it. This requires an advance knowledge of the maximum expected values for various variables in the problem. While it is possible to estimate the pessimistic maxima in a large number of cases, the situation in which these could not be easily evaluated may not altogether be ruled out in more complicated problems. Problem setting in such situations involves a trial and error approach. There are, on the other hand, occasions where this type of adaptability of the analog computer to a trial and error approach in problem solving becomes advantageous in rendering the computer useful in the design of a class of systems. Finally, there is an apparent limit to the accuracy with which the computer can furnish a solution, and most present-day analog computers are useful when errors of the order of one to two percent are allowable. This error is partly attributable to the accuracy with which the continuous variable constituting the data can be sensed and displayed. Besides, the analog circuits are liable to drift, drift being inherent in most electronic circuits. The drift during the solution of a single relatively simple equation may be negligible. But it is certainly undesirable in simulations lasting over an extended period.

Some of the above limitations are factual and are to be accepted, while others are more likely to be overcome with the developments in circuitry using improved techniques. Current effort is directed towards developing high-speed machines using faster operating and more stable electronic circuits. There is room for further improvement in terms of the attainable accuracy. Interest is also being shown in the simulation of complex systems using a "hybrid computer," which combines the speed and versatility of the electronic analog computer and the accuracy of the digital compter. These are to mention but only a few trends in current developmental and research activity concerned with this class of computers. In conclusion, it is appropriate to expect that, in spite of the various general limitations, the analog computer will continue to be considered an increasingly important tool in analysis.

Acknowledgment

The author would like to thank Dr. D. N. Norris, Jr., for his many helpful criticisms in preparing this revision.

References

[1] W. J. Karplus and W. W. Soroka, *Analog Methods—Computation and Simulation*. New York: McGraw-Hill, 1959.
[2] L. Levine, *Methods for Solving Engineering Problems Using Analog Computers*. New York: McGraw-Hill, 1964.
[3] G. A. Korn and T. M. Korn, *Electronic Analog Computers*, 2nd ed. New York: McGraw-Hill, 1956.
[4] C. L. Johnson, *Analog Computer Techniques*, 2nd ed. New York: McGraw-Hill, 1963.
[5] L. Marton, Ed., *Advances in Electronics and Electron Physics*, vol. 11. New York: Academic Press, 1959, pp. 225–283.
[6] S. Seely, *Electronic Engineering*. New York: McGraw-Hill, 1956, pp. 161–189.
[7] *Ibid.*, pp. 169–171.
[8] J. N. Warfield, *Electronic Analog Computers*. Englewood Cliffs, N. J.: Prentice-Hall, 1959.
[9] *Shock and Vibration Handbook*, C. M. Harris and C. E. Crede, Eds. New York: McGraw-Hill, 1961.
[10] S. Tse, E. Morse, and T. Hinkle, *Mechanical Vibrations*. London: Prentice-Hall International, 1963.
[11] W. W. Seto, *Theory and Problems of Mechanical Vibrations*. New York: Schaum Publishing Co., 1964.
[12] J. J. D'Azzo and C. H. Houpis, *Feedback Control System Analysis and Synthesis*. New York: McGraw-Hill, 1960.
[13] W. R. Bennett, "A general review of linear varying parameter and nonlinear circuit analysis," *Proc. IRE*, vol. 38, pp. 259–263, March 1950.
[14] L. Bush and P. Orlando, "A perturbation technique for analog computers," *IRE Trans. on Electronic Computers*, vol. EC-8, pp. 218–221, June 1959.
[15] N. W. McLachlan, *Ordinary Nonlinear Differential Equations in Engineering and Physical Sciences*, 2nd ed. Oxford, England: Clarendon Press, 1956.
[16] L. Levine, *op. cit.*, pp. 111–152.
[17] N. W. McLachlan, *op. cit.*, pp. 160–166. See also J. N. Warfield, *op. cit.*, p. 29.
[18] N. W. McLachlan, *op. cit.*, pp. 120–121.
[19] *Ibid.*, pp. 140–141.
[20] D. M. Mackay and M. E. Fisher, *Analog Computing at Ultra-High Speed*. London: Chapman and Hall, 1962.
[21] R. A. Smith, *Wave Mechanics of Crystalline Solids*. London: Chapman and Hall, 1963.
[22] G. E. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*. New York: Wiley, 1964.
[23] For a list of references on this topic, see D. M. MacKay and M. E. Fisher, *op. cit.*, pp. 289–292.
[24] L. Marton, *op. cit.*, pp. 225–283.