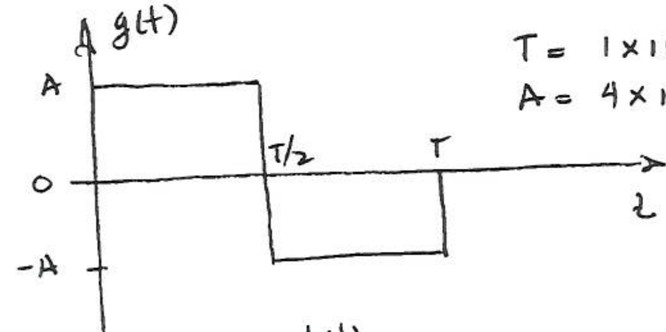
3) A communication system transmits binary information using the antipodal pulses $g(t)$ or $-g(t)$, where $g(t)$ is shown below

a) Find the impulse response of the matched filter.

b) Sketch the output of the matched filter when the pulse $g(t)$ is applied to the input. Show all important details.

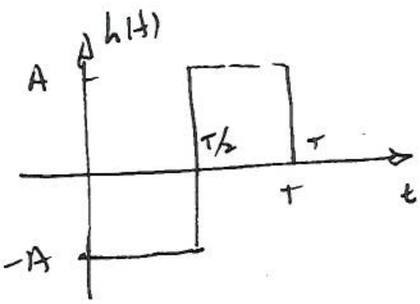
c) Suppose that two-sided noise power spectral density is 10^{-12} watts/Hz. What is the bit energy-to-noise ratio?

d) What is the probability of bit error. You can leave your answer in terms of a Q-function.

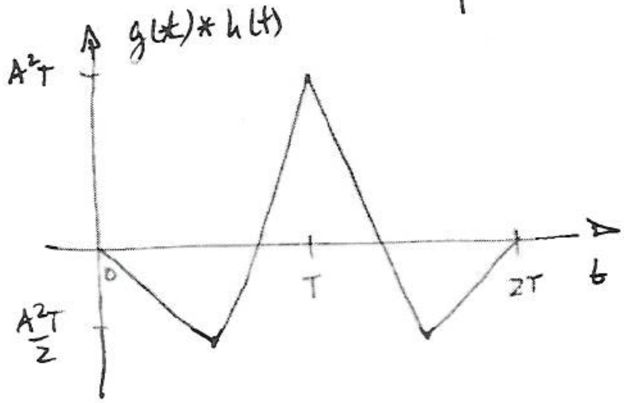


$T = 1 \times 10^{-6}$
$A = 4 \times 10^{-3}$

a) $h(t) = g(T-t)$



b)



c) $\frac{E_b}{N_o} = \frac{A^2 T}{N_o} = \frac{16 \times 10^{-6} \times 10^{-6}}{2 \times 10^{-12}} = 8 = 9.03\ dB$

d) $P_b = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) = Q(4)$

1) Random Processes: Suppose that $X(t)$ and $Y(t)$ are wide-sense stationary random processes with means μ_X and μ_Y, autocorrelation functions $\phi_{XX}(\tau)$ and $\phi_{YY}(\tau)$, power spectral densities $\Phi_{XX}(f)$ and $\Phi_{YY}(f)$, cross-correlation function $\phi_{XY}(\tau)$ and cross power spectral density $\Phi_{XY}(f)$.

Consider the sum process

$$Z(t) = X(t) + Y(t)$$

a) 4 marks: Derive an expression for the autocorrelation function of $Z(t)$ in terms of $\phi_{XX}(\tau)$, $\phi_{YY}(\tau)$ and $\phi_{XY}(\tau)$.

b) 3 marks: Suppose that $X(t)$ and $Y(t)$ are *uncorrelated* meaning that their crosscovariance function $\mu_{XY}(\tau) = \phi_{XY}(\tau) - \mu_X\mu_Y = 0$. What is the autocorrelation function of $Z(t)$?

c) 2 marks: Suppose further that $X(t)$ and $Y(t)$ are uncorrelated and $\mu_X = \mu_Y = 0$. What is the autocorrelation function of $Z(t)$?

d) 1 marks: What is the power spectral density of $Z(t)$ in part c)?

a)
$$\begin{aligned}
\phi_{zz}(\tau) &= E\left[z(t) z(t+\tau) \right] \\
&= E\left[(x(t) + y(t))(x(t+\tau) + y(t+\tau)) \right] \\
&= E\left[x(t)x(t+\tau) + y(t)x(t+\tau) + x(t)y(t+\tau) + y(t)y(t+\tau) \right] \\
&= E\left[x(t)x(t+\tau)\right] + E\left[y(t)y(t+\tau)\right] + E\left[x(t)y(t+\tau)\right] + E\left[y(t)y(t+\tau)\right] \\
&= \phi_{xx}(\tau) + \phi_{yx}(\tau) + \phi_{xy}(\tau) + \phi_{yy}(\tau)
\end{aligned}$$

use $\phi_{yx}(\tau) = \phi_{xy}(-\tau)$

$$= \phi_{xx}(\tau) + \phi_{xy}(-\tau) + \phi_{xy}(\tau) + \phi_{yy}(\tau)$$

b) $\phi_{zz}(\tau) = \phi_{xx}(\tau) + \phi_{yy}(\tau) + 2\mu_x\mu_y$

c) $\phi_{zz}(\tau) = \phi_{xx}(\tau) + \phi_{yy}(\tau)$

d) $\Phi_{zz}(f) = \Phi_{xx}(f) + \Phi_{yy}(f)$

1) Consider a wide sense stationary random process $X(t)$ having mean μ_X and autocorrelation function $\phi_{XX}(\tau)$ and power spectrum $S_{XX}(f)$. The random process $X(t)$ is used to construct another random process $Z(t)$ as follows:

$$Z(t) = X(t) - X(t - 2T)$$

where all quantities are real valued.

a) 1 mark What is the output mean μ_Z?

b) 3 marks What is the autocorrelation of $Z(t)$, $\phi_{ZZ}(\tau)$, in terms of $\phi_{XX}(\tau)$?

c) 2 marks What is the output power spectrum $S_{ZZ}(f)$?

d) 3 marks What is the cross-correlation of $X(t)$ and $Z(t)$, $\phi_{XZ}(\tau)$, in terms of $\phi_{XX}(\tau)$

d) 1 mark If the input process $X(t)$ was a Gaussian random process, is $Z(t)$ a Gaussian random process?

a) $$\mu_Z = E[Z(t)] = E[X(t) - X(t-2T)]$$
$$= E[X(t)] - E[X(t-2T)] = \mu_X - \mu_X = 0$$

b) $$\phi_{ZZ}(\tau) = E[Z(t)\,Z(t+\tau)]$$
$$= E[(X(t) - X(t-2T))(X(t+\tau) - X(t-2T+\tau))]$$
$$= E[X(t)X(t+\tau)] - E[X(t-2T)X(t+\tau)]$$
$$- E[X(t)X(t-2T+\tau)] + E[X(t-2T)X(t-2T+\tau)]$$
$$= \phi_{xx}(\tau) - \phi_{xx}(\tau+2T) - \phi_{xx}(\tau-2T) + \phi_{xx}(\tau)$$
$$= 2\phi_{xx}(\tau) - \phi_{xx}(\tau+2T) - \phi_{xx}(\tau-2T)$$

c) $$S_{ZZ}(f) = 2S_{xx}(f) - S_{xx}(f)e^{-j4\pi fT} - S_{xx}(f)e^{j4\pi fT}$$
$$= 2S_{xx}(f)\left(1 - \frac{e^{j4\pi fT} + e^{-j4\pi fT}}{2}\right)$$
$$= 2(1 - \cos 4\pi fT)\,S_{xx}(f)$$
$$= 4\sin^2(2\pi fT)\,S_{xx}(f)$$

d) $$\phi_{xz}(\tau) = E\left[x(t)\,z(t+\tau)\right]$$
$$= E\left[x(t)\left(x(t+\tau) - x(t+\tau-2T)\right)\right]$$
$$= E\left[x(t)x(t+\tau)\right] - E\left[x(t)x(t+\tau-2T)\right]$$
$$= \phi_{xx}(\tau) - \phi_{xx}(\tau - 2T)$$

e) Yes, at any time t_1

$$z(t_1) = x(t_1) + x(t_1 - 2T)$$

Gaussian

sum of Gaussian random variables is a Gaussian random variable.

1) Random Processes: Suppose that the inputs $X(t)$ and $Y(t)$ to a multiplier are independent random processes with power spectral densities

$$S_X(f) = 5\mathrm{rect}\left(\frac{f}{10}\right)$$
$$S_Y(f) = 2\mathrm{rect}\left(\frac{f}{6}\right)$$

a) 7 marks: Compute and sketch the power spectral density at the output of the multiplier.

b) 3 marks: What is the total power at the output?

a) We have seen from Homework that

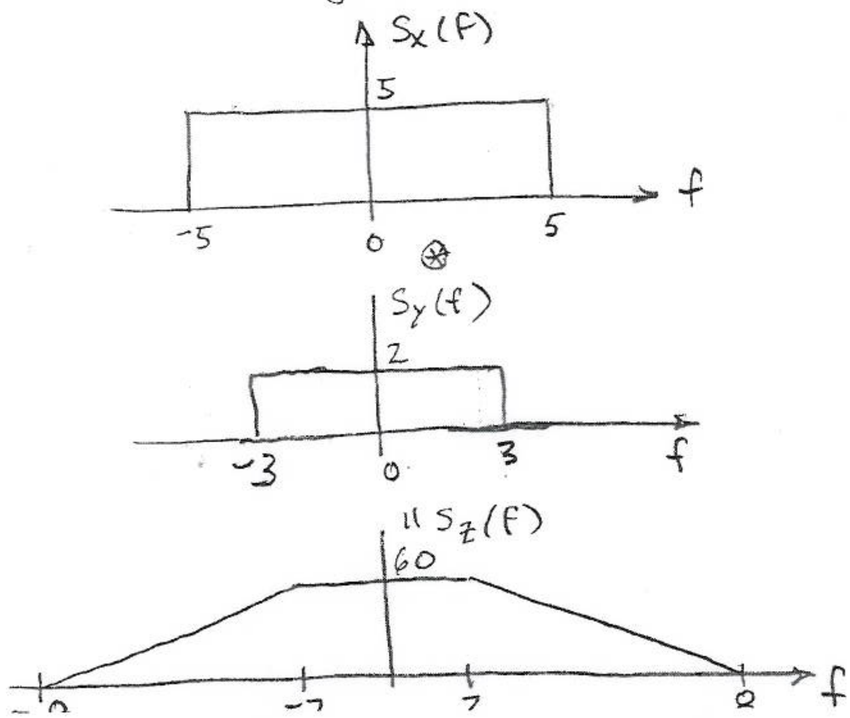
$Z(t) = X(t)Y(t)$ has autocorrelation function

$$\phi_{ZZ}(\tau) = \phi_{XX}(\tau)\,\phi_{YY}(\tau)$$

Hence, $S_Z(f) = S_X(f) * S_Y(f)$, where

$*$ denotes convolution.

You can use graphical convolution



b) total power

$$P_t = \int_{-\infty}^{\infty} S_z(f)\,df$$

$$= \frac{1}{2}(6)(60) + (4)(60) + \frac{1}{2}(6)(60)$$

$$= 600 \text{ watts}$$

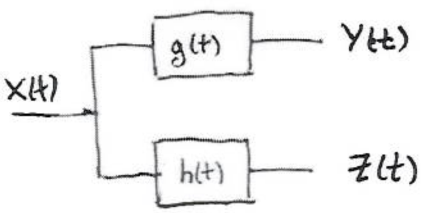
2) The random process $X(t)$ with mean μ_X and autocorrelation function $\phi_{XX}(\tau)$ is applied to the filters $g(t)$ and $h(t)$ as shown below. The output processes are $Y(t)$ and $Z(t)$.

a) Find the output means μ_Y and μ_Z in terms of μ_X.

b) Find the cross-correlation function

$$\phi_{YZ}(\tau) = \mathrm{E}[Y(t)Z(t+\tau)]$$

in terms of $\phi_{XX}(\tau)$, $g(t)$ and $h(t)$.



a) $\mu_Y = G(0)\mu_X \qquad \mu_Z = H(0)\mu_X$

b) $$\phi_{YZ}(\tau) = E[Y(t)Z(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} g(\alpha)X(t-\alpha)\,d\alpha \int_{-\infty}^{\infty} h(\beta)X(t+\tau-\beta)\,d\beta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha)h(\beta)\,E[X(t-\alpha)X(t+\tau-\beta)]\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha)h(\beta)\,\phi_{xx}(\tau-\beta+\alpha)\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty} h(\beta)\int_{-\infty}^{\infty} g(\alpha)\phi_{xx}(\tau+\alpha-\beta)\,d\alpha\,d\beta$$

$$= \left\{\int_{-\infty}^{\infty} g(\alpha)\phi_{xx}(\tau+\alpha)\,d\alpha\right\} * h(\tau)$$

$$= g(-\tau) * \phi_{xx}(\tau) * h(\tau)$$

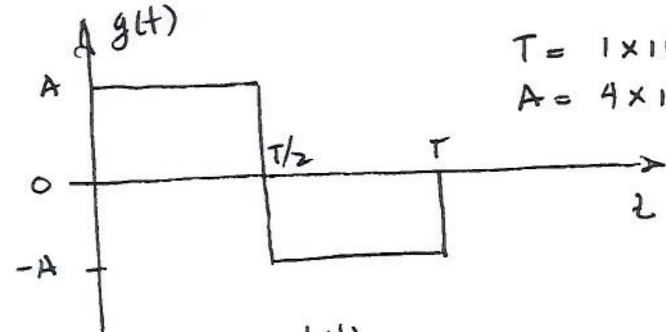
3) A communication system transmits binary information using the antipodal pulses $g(t)$ or $-g(t)$, where $g(t)$ is shown below

a) Find the impulse response of the matched filter.

b) Sketch the output of the matched filter when the pulse $g(t)$ is applied to the input. Show all important details.

c) Suppose that two-sided noise power spectral density is 10^{-12} watts/Hz. What is the bit energy-to-noise ratio?

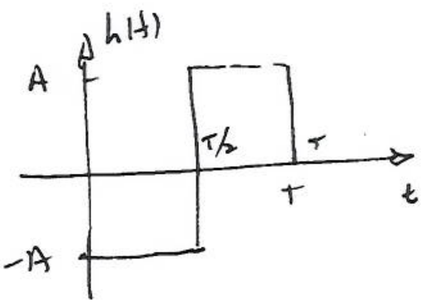
d) What is the probability of bit error. You can leave your answer in terms of a Q-function.



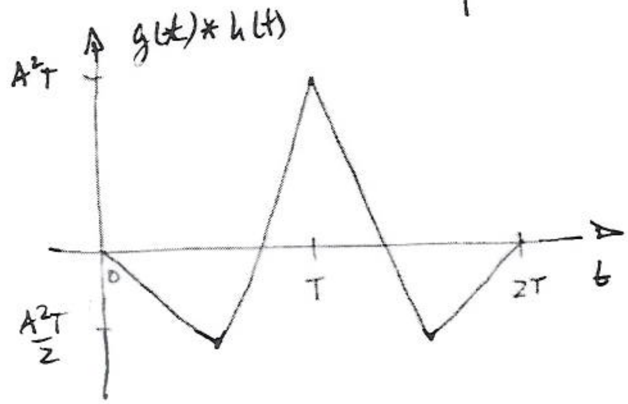
$T = 1 \times 10^{-6}$

$A = 4 \times 10^{-3}$

a) $h(t) = g(T-t)$



b)



c) $\frac{E_b}{N_o} = \frac{A^2T}{N_o} = \frac{16\times10^{-6}\times10^{-6}}{2\times10^{-12}} = 8 = 9.03\text{ dB}$

d) $P_b = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) = Q(4)$

2) Consider a binary communication system that transmits information using the pulse

$$g(t) = A[-u(t) + 2u(t - T/2) - u(t - T)]$$

according to the mapping rule

$$\begin{aligned} \text{“0”} &\longrightarrow -g(t) \\ \text{“1”} &\longrightarrow +g(t) \end{aligned}$$

The “0”s and “1”s are transmitted with equal probability, and the channel is an AWGN channel, with a two-sided noise power spectral density of $N_o/2$ watts/Hz.

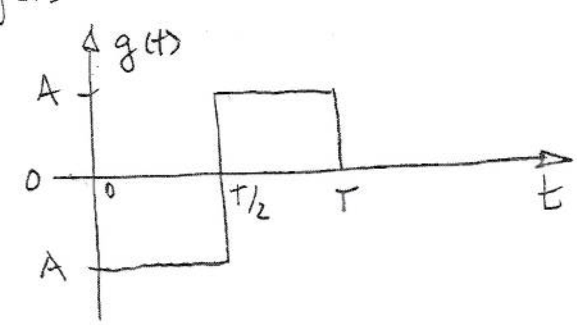
a) 3 marks Determine and sketch the filter $h(t)$ that is matched to $g(t)$.

b) 3 marks Determine and sketch the overall pulse $p(t) = g(t) * h(t)$ for the filter you found in part (a), labelling all important points.

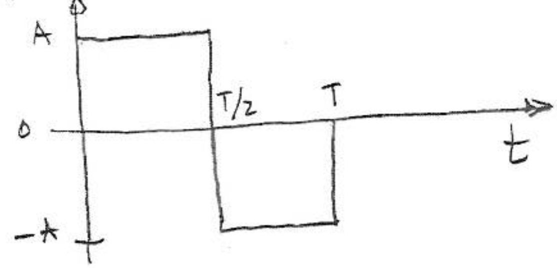
c) 2 marks Determine the pulse energy, E, and the energy per bit, E_b.

d) 2 marks What is the probability of bit error in terms of E_b/N_o?

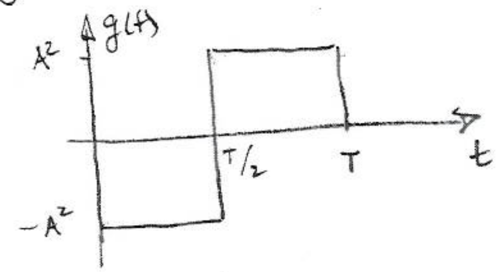
a) Pulse $g(t)$ is

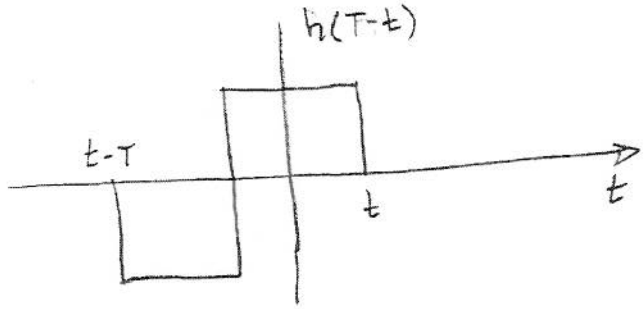


$h(t) = g(T-t)$



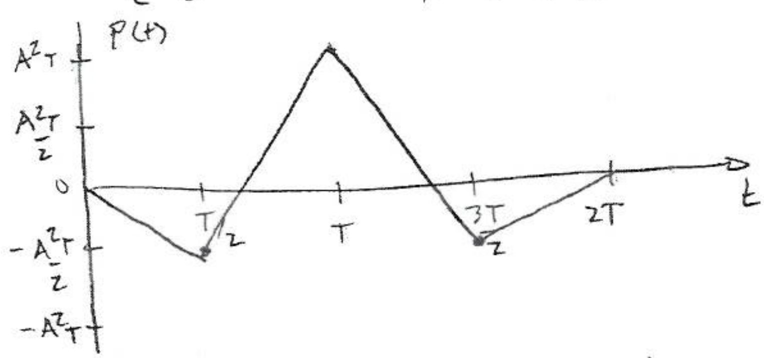
b) You can use graphical convolution





You can verify

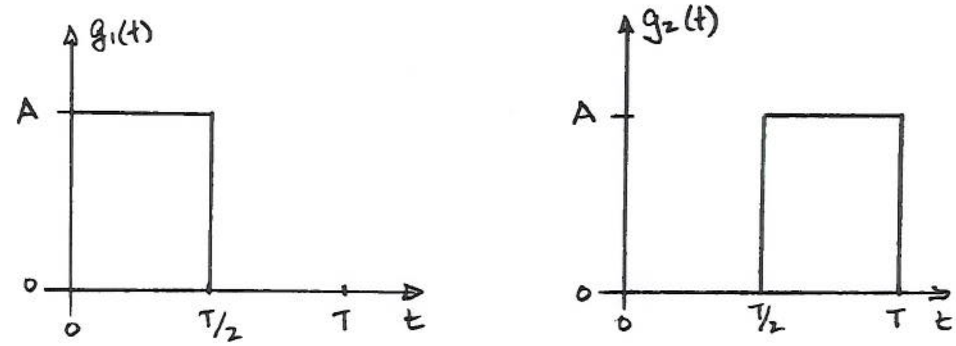
$$p(t) = \begin{cases} 0, & t \le 0 \\ -A^2 t, & 0 \le t \le T/2 \\ 3A^2 t - 2A^2 T, & T/2 \le t \le T \\ -3A^2 t + 4A^2 T, & T \le t \le \frac{3T}{2} \\ -2A^2 T + A^2 t, & \frac{3T}{2} \le t \le 2T \\ 0, & t \geq 2 \end{cases}$$



c) $E = A^2 T$, $E_b = \frac{1}{2}E + \frac{1}{2}E = E$

d) $P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

2) Consider the pair of pulses $g_1(t)$ and $g_2(t)$ shown in the figure below.

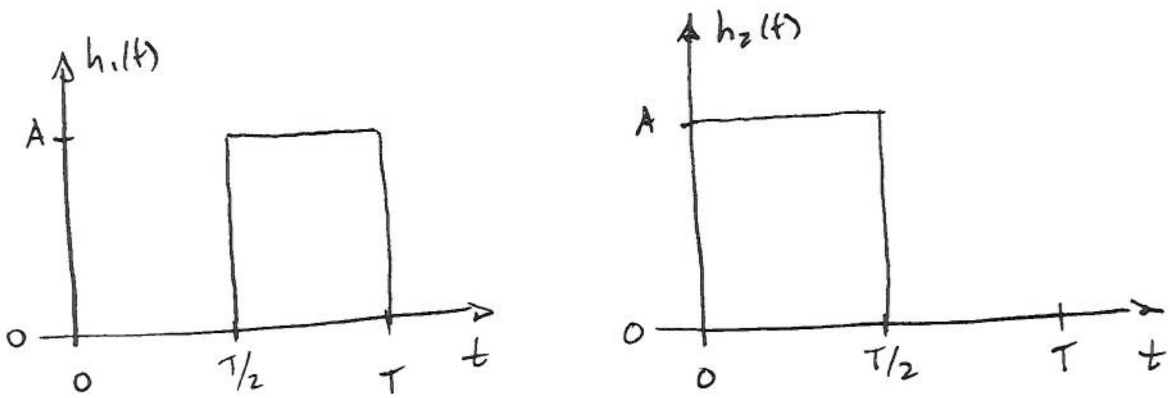


a) 4 marks Sketch the matched filters for these two pulses, $h_1(t)$ and $h_2(t)$.

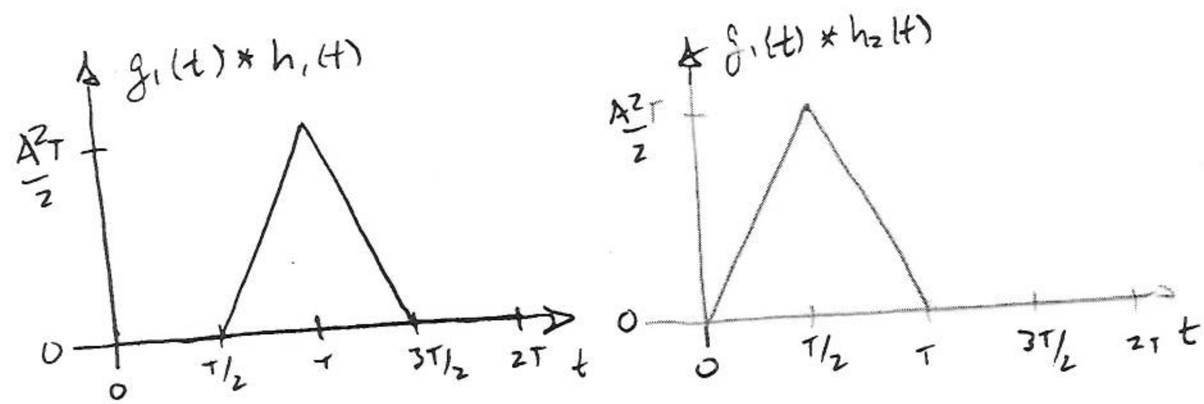
b) 3 marks Sketch the waveforms at the output of the filters $h_1(t)$ and $h_2(t)$ if the input is $g_1(t)$.

c) 3 marks Sketch the waveforms at the output of the filters $h_1(t)$ and $h_2(t)$ if the input is $g_1(t) + g_2(t)$.

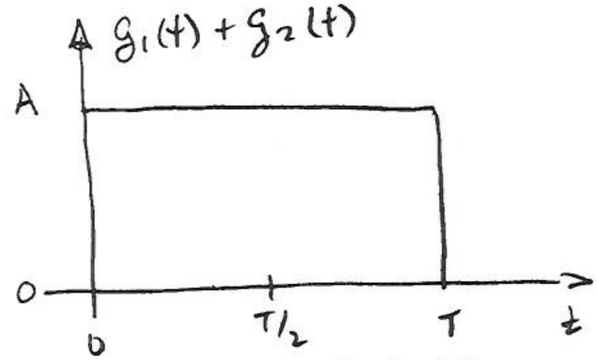
a)



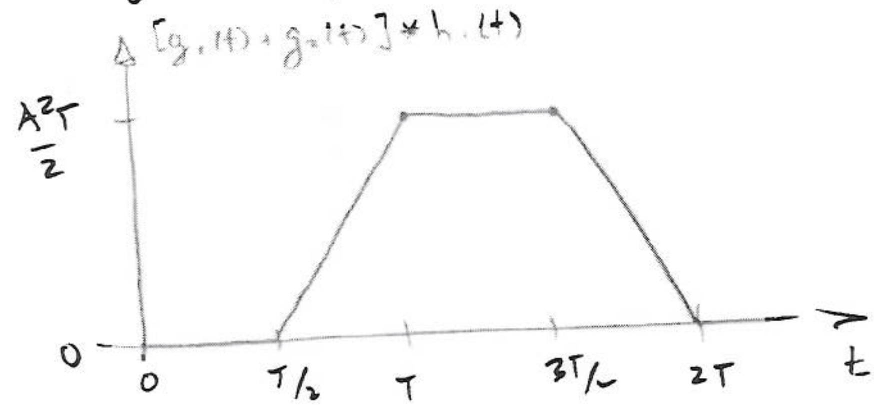
b)



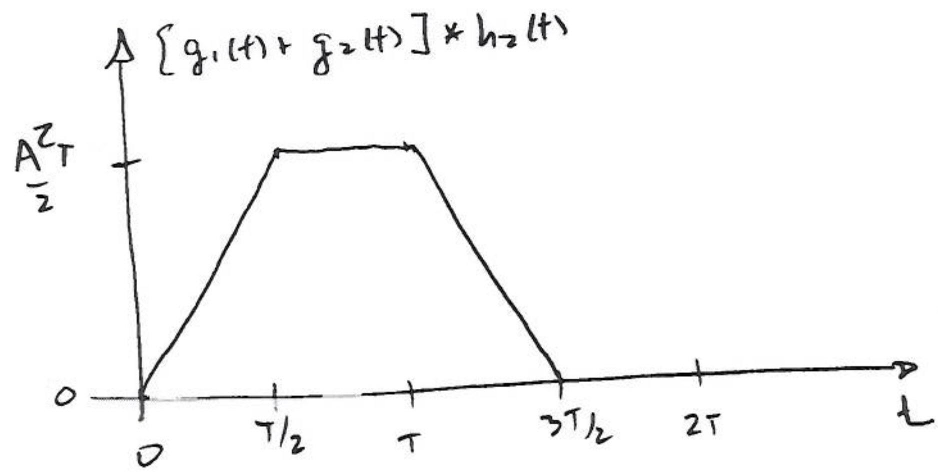
c)



g1(t) + g2(t)
A
0
0
T/2
T
t

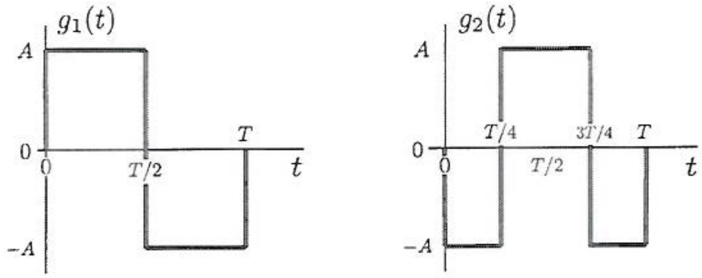


[g1(t) + g2(t)] * h1(t)
A²T/2
0
0
T/2
T
3T/2
2T
t



[g1(t) + g2(t)] * h2(t)
A²T/2
0
0
T/2
T
3T/2
2T
t

2) Matched Filters: Consider the two pulses $g_1(t)$ and $g_2(t)$ shown below.

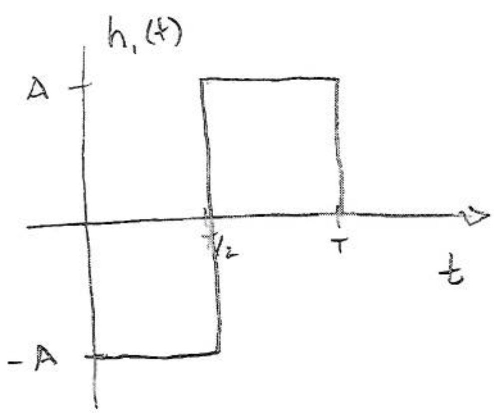


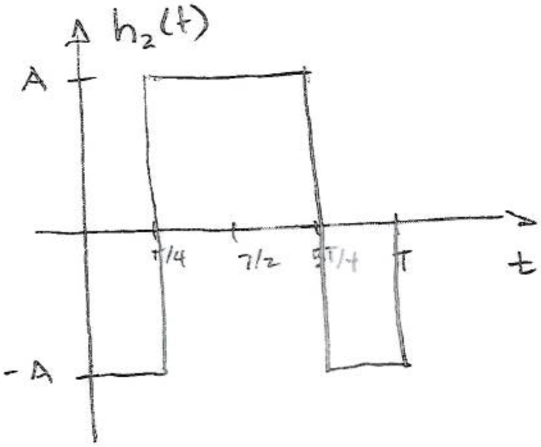
a) 4 marks Determine and sketch the corresponding matched filters $h_1(t)$ and $h_2(t)$.

b) 3 marks Sketch the outputs of the two matched filters when the pulse $g_1(t)$ is applied at their input.

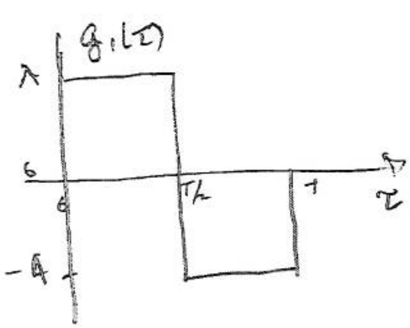
c) 3 marks Sketch the outputs of the two matched filters when the pulse $g_2(t)$ is applied at their input.

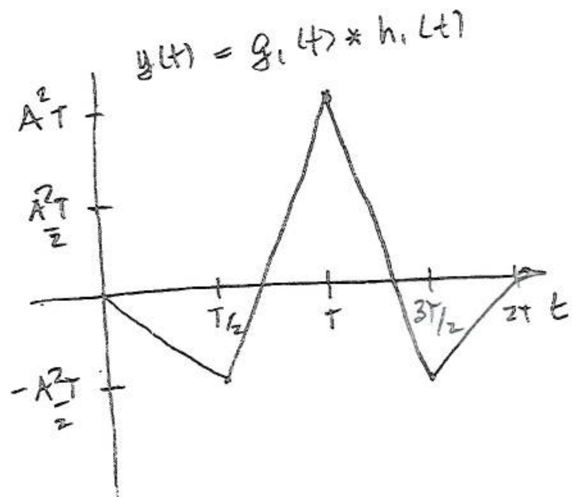
a)

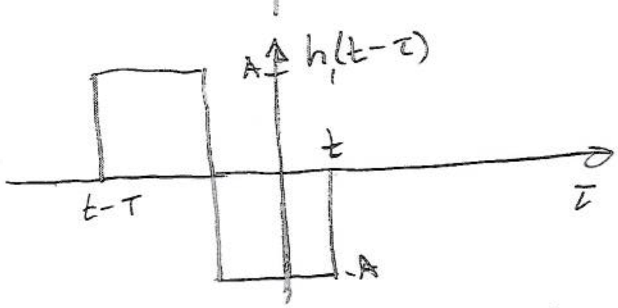




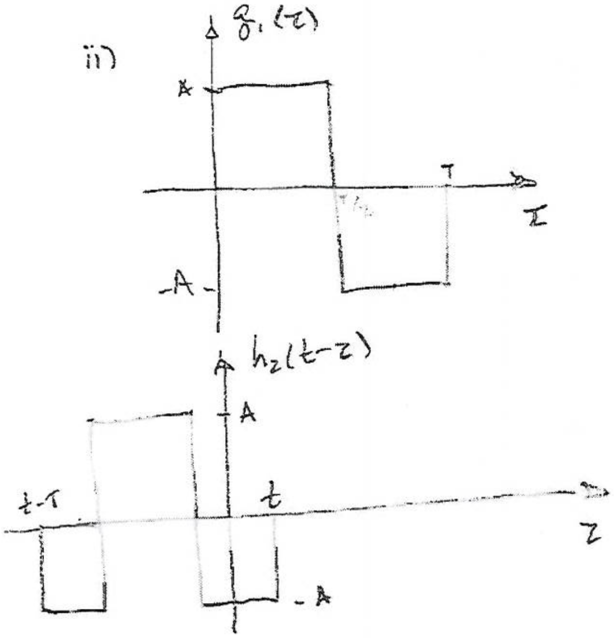
b) i)



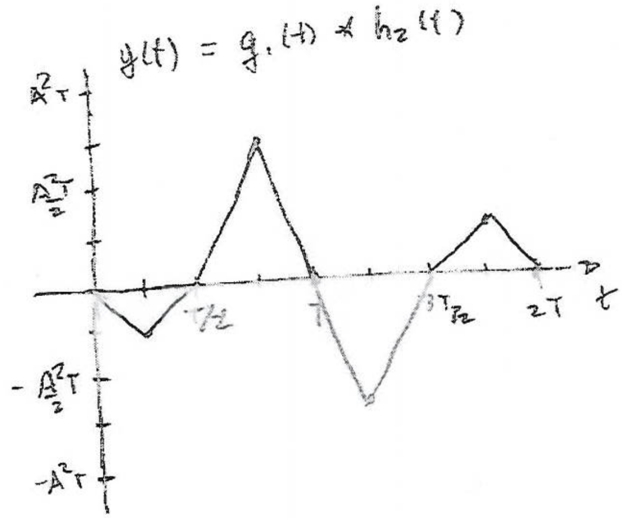




ii)

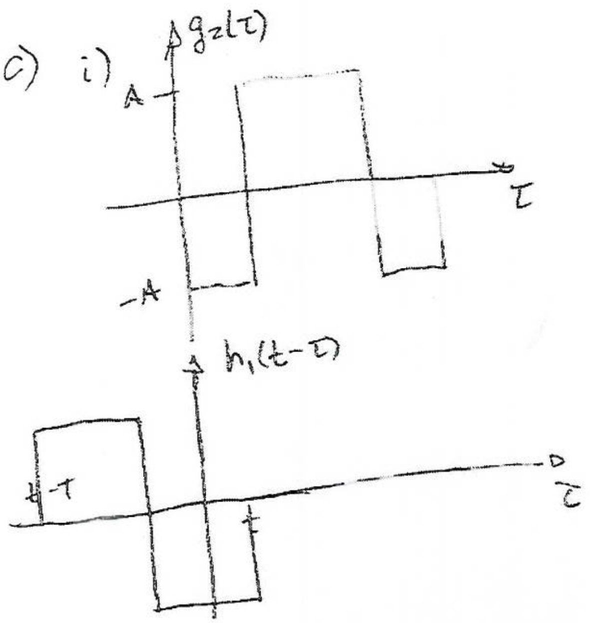


g1(τ)
A
-A
T/2
T
τ
h2(t-τ)
A
t-T
t
-A
τ

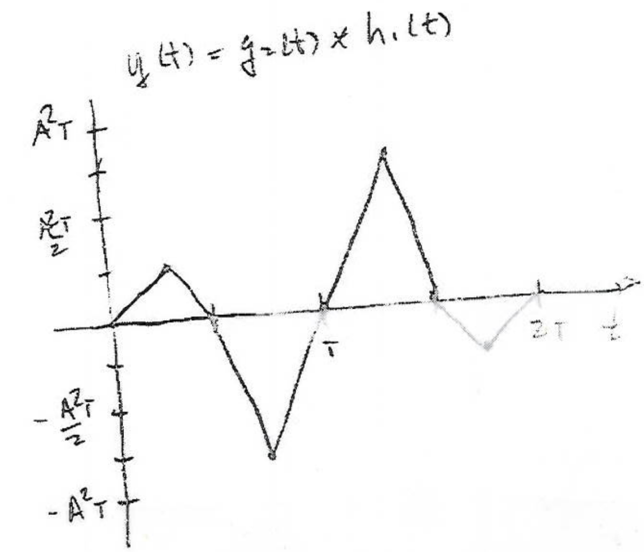


y(t) = g1(t) * h2(t)
A²T
A²T/2
-A²T/2
-A²T
T/2
T
3T/2
2T
t

c) i)

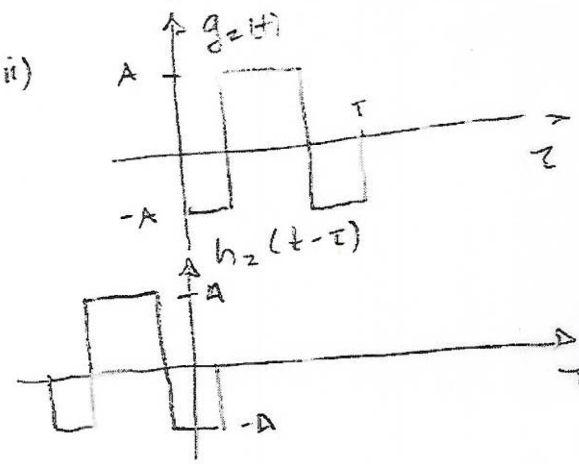


g2(τ)
A
-A
τ
h1(t-τ)
t-T
t
τ

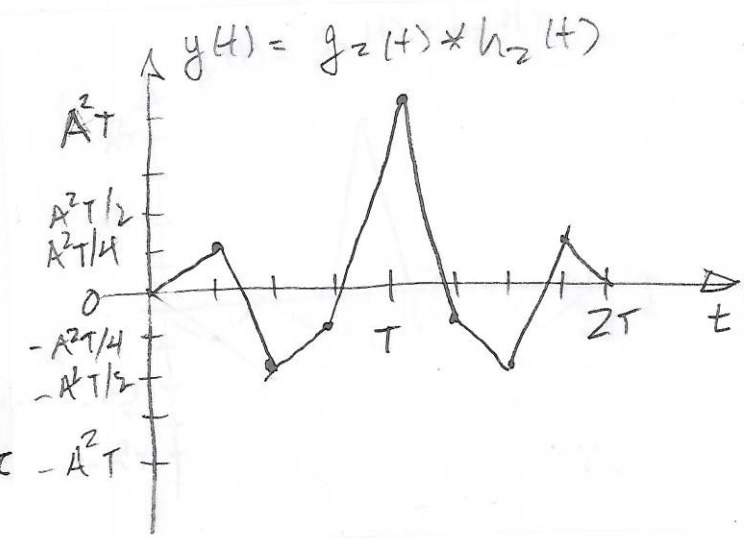


y(t) = g2(t) * h1(t)
A²T
A²T/2
-A²T/2
-A²T
T
2T
t

ii)



g2(t)
A
-A
T
τ
h2(t-τ)
A
-A
τ



y(t) = g2(t) * h2(t)
A²T
A²T/2
A²T/4
0
-A²T/4
-A²T/2
-A²T
T
2T
t