Solutions.

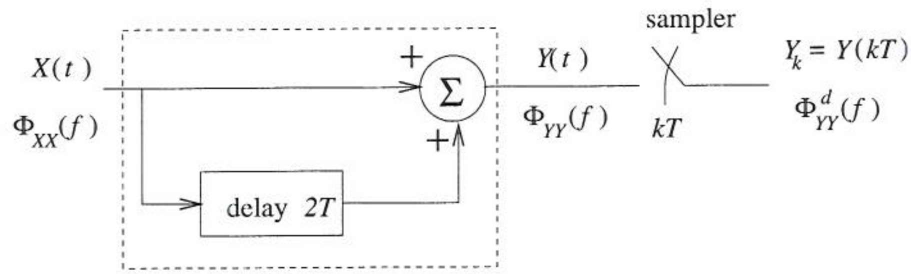
GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING
Final Examination - Fall 2012
EE 4601: Communication Systems

Aids Allowed: Course textbook
one $8\frac{1}{2} \times 11$ two-side crib sheet
Attempt all questions
All questions are of equal value

DATE: Wednesday December 12, 2012.

TIME: 11:30am - 2:20pm
INSTRUCTOR: Professor G.L. Stüber

1) Random Processes: Consider the system shown below. The input $X(t)$ to the filter $h(t) \leftrightarrow H(f)$ is a wide-sense stationary random process with power density spectrum $\Phi_{XX}(f)$.



filter $h(t) \longleftrightarrow H(f)$

a) (**5 points**) Find the power spectral density $\Phi_{YY}(f)$ of the random process $Y(t)$ at the output of the filter $h(t)$ in terms of $\Phi_{XX}(f)$.

b) (**3 points**) What frequency components cannot be present in the filter output $Y(t)$?

c) (**2 points**) Find the power spectral density $\Phi^d_{YY}(f)$ of the discrete-time random process Y_k at the output of the sampler.

a) $$\Phi_{YY}(f) = |H(f)|^2 \Phi_{XX}(f)$$

$$H(f) = 1 + e^{-j4\pi fT}$$

$$= 2e^{-j2\pi fT}\left(\underline{e^{j2\pi fT} + e^{-j2\pi fT}}\right)$$

$$= 2\cos(2\pi fT)\, e^{-j2\pi f\overset{2}{T}}$$

$$|H(f)|^2 = 4\cos^2(2\pi fT)$$

$$\Phi_{YY}(f) = 4\cos^2(2\pi fT)\,\Phi_{XX}(f)$$

b) $2\pi fT = k\frac{\pi}{2}$ $\quad f = \frac{k}{4T}$, k odd integer

c) Due to sampling, we have

$$\phi_\delta(t) = \sum_k \phi_{YY}(kT)\,\delta(t - nT)$$

$$\Phi^d_{YY}(f) = f_s \sum_m \Phi_{YY}(f - mf_s)$$

$$= \frac{1}{T} \sum_m \Phi_{YY}\left(f - \frac{m}{T}\right)$$

$$= \frac{1}{T} \sum_m \left| H\left(f - \frac{m}{T}\right)\right|^2 \Phi_{XX}\left(f - \frac{m}{T}\right)$$

$$= \frac{4}{T} \sum_m \cos^2\left(2\pi (f - m/T) T\right) \Phi_{XX}\left(f - \frac{m}{T}\right)$$

$$= \frac{4}{T} \sum_m \cos^2\left(2\pi (fT - m)\right) \Phi_{XX}\left(f - \frac{m}{T}\right)$$

2) ISI Channels: Suppose that the frequency response characteristic of a channel can be approximated by

$$H(f) = 1 + \alpha \cos(2\pi f t_o) \quad ,$$

where $|\alpha| < 1$. A pulse $g(t)$ is passed through the channel.

a) (**5 points**) Find the signal $y(t) = g(t) * h(t)$ at the output of the channel.

b) (**3 points**) Suppose the received signal $y(t)$ is passed through a filter matched to $g(t)$, such that $p(t) = g(t) * g(-t)$. Determine the sampled output of the matched filter $q_k = q(kT)$, at time $t = kT, k = 0, \pm 1, \pm 2, \ldots$, where $q(t) = y(t) * g(-t) = g(t) * h(t) * g(-t) = p(t) * h(t)$.

c) (**2 points**) What is the overall sampled pulse $q_k = q(kT)$ if $t_o = T$?

a)

$$H(f) = 1 + \alpha \cos(2\pi f t_o)$$

$$= 1 + \frac{\alpha e^{j2\pi f t_o}}{2} + \frac{\alpha e^{-j2\pi f t_o}}{2}$$

$$h(t) = \delta(t) + \frac{\alpha}{2}\delta(t+t_o) + \frac{\alpha}{2}\delta(t-t_o)$$

$$y(t) = g(t) * h(t)$$

$$= g(t) + \frac{\alpha}{2} g(t+t_o) + \frac{\alpha}{2} g(t-t_o)$$

b)

$$q(t) = g(t) * h(t) * g(-t)$$

$$= p(t) * h(t)$$

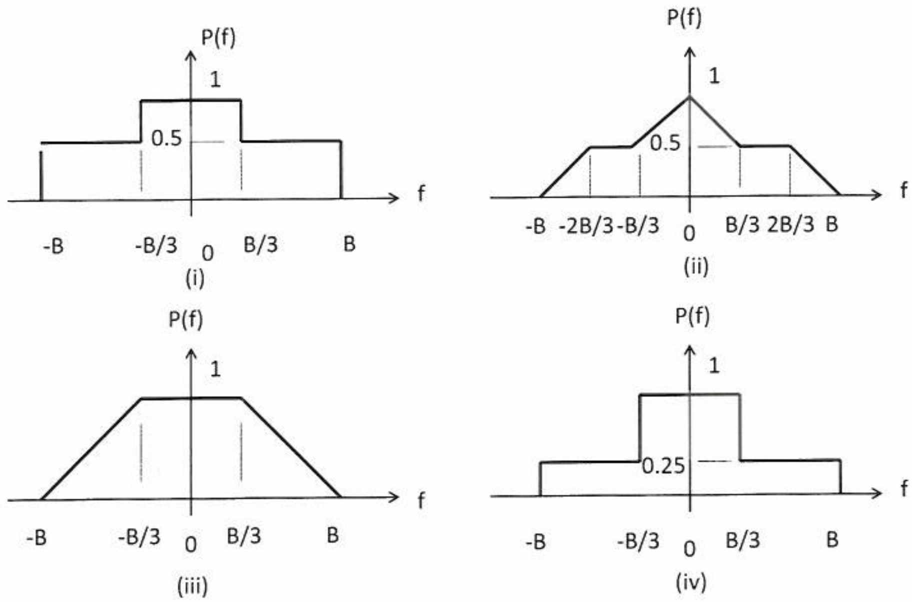
$$= p(t) + \frac{\alpha}{2} p(t+t_o) + \frac{\alpha}{2} p(t-t_o)$$

$$q_k = p(kT) + \frac{\alpha}{2} p(kT+t_o) + \frac{\alpha}{2} p(kT-t_o)$$

c) $$q_k = \frac{\alpha}{2} p_{k-1} + p_k + \frac{\alpha}{2} p_{k+1}$$

3) Intersymbol Interference: For each of the overall system responses in the figure below, $P(f) = G(f)C(f)H(f)$, where $G(f)$ is the transmit filter, $C(f)$ is the channel, $H(f)$ is the receiver filter.

a) (**10 points**) For each channel, determine if one can signal with zero ISI, and if so determine what the baud rate R is.



i) Yes, Choose $\frac{1}{2T_b} = \frac{2B}{3}$ Hz

$R_b = \frac{4}{3}B$ bits/s.

ii) Yes Choose $\frac{1}{2T_b} = \frac{B}{2}$ Hz

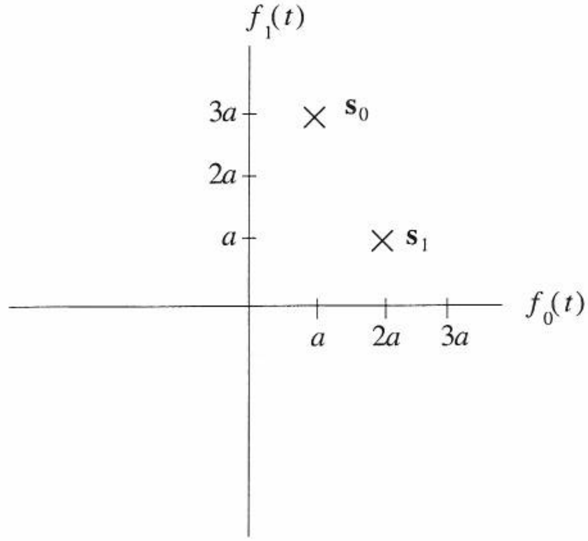
$R_b = B$ bits/s

iii) Yes, Choose $\frac{1}{2T_b} = \frac{2B}{3}$ Hz

$R_b = \frac{4}{3}B$ bits/s

iv) No, folded spectrum not flat.

4) Signal Space: Consider a binary communication system whose signals are represented in the signal-space diagram shown below.



a) **(3 points)** Calculate the correlation coefficient ρ_{01} between the two signal waveforms $s_0(t)$ and $s_1(t)$.

b) **(3 points)** What is the average energy per information bit, E_b?

c) **(4 points)** Assuming an AWGN channel with a two-sided noise power spectral density of $N_o/2$ watts/Hz and minimum distance decisions, what is the probability of bit error in terms of E_b/N_o?

a)

$$\underline{s}_0 = (a, 3a)$$
$$\underline{s}_1 = (2a, a)$$

$$\rho_{01} = \frac{\underline{s}_0 \cdot \underline{s}_1}{\|\underline{s}_0\| \, \|\underline{s}_1\|}$$

$$= \frac{2a^2 + 3a^2}{\sqrt{10}\,a \cdot \sqrt{5}\,a}$$

$$= \frac{5}{5\sqrt{2}} = 1/\sqrt{2} = 0.7071$$

b)

$$E_b = \frac{10a^2 + 5a^2}{2} = \frac{15a^2}{2}$$

c) We have

$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

But $d_{12}^2 = a^2 + 4a^2 = 5a^2$

and $a^2 = \frac{2E_b}{15}$

$$\Rightarrow d_{12}^2 = \frac{10E_b}{15} = \frac{2E_b}{3}$$

$$P_b = Q\left(\sqrt{\frac{1}{3}\frac{E_b}{N_0}}\right)$$

5) Error Probability: Consider the following set of four signal vectors.

$$\begin{aligned}
\mathbf{s}_1 &= \sqrt{E/6}\ (-1,-1,-1,+1,+1,+1)\\
\mathbf{s}_2 &= \sqrt{E/6}\ (-1,+1,-1,-1,+1,+1)\\
\mathbf{s}_3 &= \sqrt{E/6}\ (+1,-1,-1,+1,-1,-1)\\
\mathbf{s}_4 &= \sqrt{E/6}\ (+1,-1,+1,-1,-1,-1)
\end{aligned}$$

The messages are transmitted over an additive white Gaussian noise channel with two-sided noise power spectral density $N_o/2$ watts/Hz.

a) (**5 points**) Suppose that minimum distance decisions are used and the messages are transmitted with equal probability. Derive a union bound on the probability of symbol error in terms of E_b/N_o.

b) (**3 points**) Derive a simple upper bound on the probability of symbol error in terms of the minimum distance between signal vectors. Express your answer in terms of E_b/N_o.

c) (**2 points**) Specify a set of time domain waveforms $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$, defined on the interval $0 \le t \le T$ that have the above signal vectors. *Note: The solution is not unique.*

a) $$P_M \le \frac{1}{M} \sum_{k=1}^{M} \sum_{j \ne k} P(\underline{s}_k, \underline{s}_j)$$

Here $M = 4$ and

$$P(\underline{s}_k, \underline{s}_j) = Q\left(\sqrt{\frac{d_{jk}^2}{2N_o}}\right)$$

$d_{12}^2 = 8E/6$

$d_{13}^2 = 12E/6$

$d_{14}^2 = 20E/6$

$d_{23}^2 = 20E/6$

$d_{24}^2 = 20E/6$

$d_{34}^2 = 8E/6$

$E_1 = E_2 = E_3 = E_4 = E$

$E_b = E/2$

or $E = 2E_b$

$$P_M \le \frac{1}{4}\Big[P(\underline{s}_1,\underline{s}_2) + P(\underline{s}_1,\underline{s}_3) + P(\underline{s}_1,\underline{s}_4)$$
$$+ P(\underline{s}_2,\underline{s}_1) + P(\underline{s}_2,\underline{s}_3) + P(\underline{s}_2,\underline{s}_4)$$
$$+ P(\underline{s}_3-\underline{s}_1) + P(\underline{s}_3,\underline{s}_2) + P(\underline{s}_3,\underline{s}_4)$$
$$+ P(s_4,\underline{s}_1) + P(\underline{s}_4,\underline{s}_2) + P(\underline{s}_4,\underline{s}_3)\Big]$$

$$= \frac{1}{4}\Big[4Q\Big(\sqrt{\frac{16}{12}\frac{E_b}{N_0}}\Big) + 2Q\Big(\sqrt{\frac{24}{12}\frac{E_b}{N_0}}\Big)$$
$$+ 6Q\Big(\sqrt{\frac{40}{12}\frac{E_b}{N_0}}\Big)\Big]$$

$$= Q\Big(\sqrt{\frac{4}{3}\frac{E_b}{N_0}}\Big) + \frac{1}{2}Q\Big(\sqrt{\frac{2E_b}{N_0}}\Big) + \frac{3}{2}Q\Big(\sqrt{\frac{10}{3}\frac{E_b}{N_0}}\Big)$$

b)

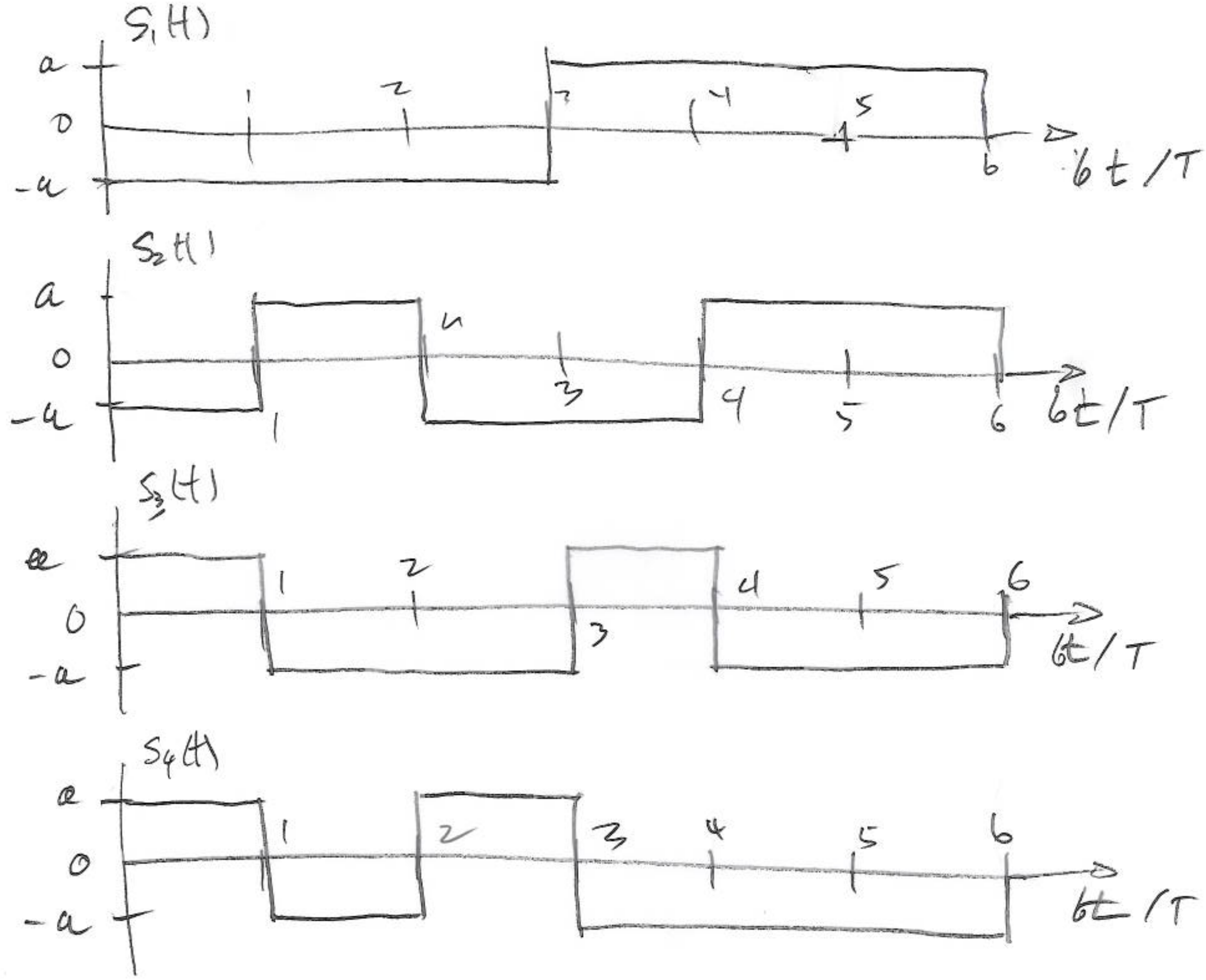
$$P_M < (M-1)\,Q\Big(\sqrt{\frac{d_{min}^2}{2N_0}}\Big)$$

$$= 3Q\Big(\sqrt{\frac{16}{12}\frac{E_b}{N_0}}\Big)$$

$$= 3Q\Big(\sqrt{\frac{4}{3}\frac{E_b}{N_0}}\Big)$$

$a = \sqrt{E/6}$

c)



$s_1(t)$
a
0
$-a$
1 2 3 4 5 6
$6t/T$
$s_2(t)$
a
0
$-a$
1 2 3 4 5 6
$6t/T$
$s_3(t)$
a
0
$-a$
1 2 3 4 5 6
$6t/T$
$s_4(t)$
a
0
$-a$
1 2 3 4 5 6
$6t/T$