EE4601
Communication Systems

Week 11

Non-Binary Signal Sets

QAM Error Probability

Error Probably Bounds

Rotations and Translations

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M-ary PAM

With M-ary Pulse Amplitude Modulation, information is transmitted in the carrier amplitude, such that the amplitude takes on one of M possible values.

During any baud interval, the transmitted waveform is

$$s_m(t) = \sqrt{\frac{2E_0}{T}} a_m \cos(2\pi f_c t), \ 0 \leq t \leq T$$

where

$$a_m \in \{\pm 1, \pm 3, \pm 5, \pm(M-1)\}$$

and E_0 is the energy of the signal with the lowest amplitude, i.e., when $a_m = \pm 1$.

Usually, $M = 2^k$ for some k, i.e., $M = 2, 4, 8, 16$, etc.

During each baud interval of length T, $k = \log_2 M$ bits are transmitted.

The baud rate $R = 1/T$ and the bit rate is $R_b = kR$.

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M-ary PAM

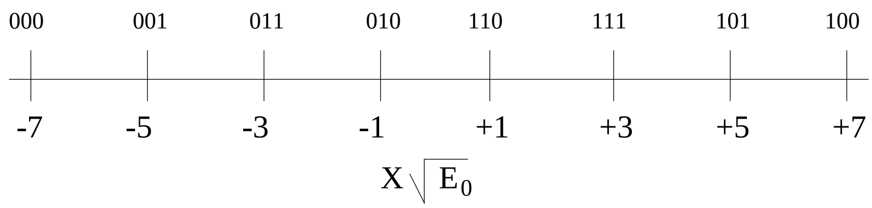
M-ary PAM signals can be expressed in terms of signal vectors. Since all the M signals are linearly dependent, there is only one basis function.

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad , \quad 0 \le t \le T$$

Then

$$s_m(t) = a_m \sqrt{E_0} f_1(t)$$

Hence, the signal-space diagram for M-ary PAM is shown below.



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M-ary QAM

Quadrature Amplitude Modulation (QAM) signals can be thought of a independent PAM on the inphase (cosine) and quadrature (sine) carrier components. During any baud interval the transmitted waveform is

$$s_m(t) = \sqrt{\frac{2E_0}{T}} \left(a_m^c \cos(2\pi f_c t) - a_m^s \sin(2\pi f_c t) \right), \ 0 \le t \le T$$

where

$$a_m^{\{c,s\}} \in \{\pm 1, \pm 3, \pm 5, \pm (M-1)\}$$

and $2E_0$ is the energy of the signal with the lowest amplitude,
i.e., when $a_m^c, a_m^s = \pm 1$.

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M-ary QAM

QAM signals can be expressed in terms of signal vectors. Since the functions $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$, with $f_c T \gg 1$, are orthogonal over the interval $(0, T)$, we have two basis functions

$$\begin{aligned} f_1(t) &= \sqrt{\frac{2}{T}} \cos 2\pi f_c t, \ 0 \le t \le T \\ f_2(t) &= -\sqrt{\frac{2}{T}} \sin 2\pi f_c t, \ 0 \le t \le T \end{aligned}$$

Then

$$s_m(t) = a_m^c \sqrt{E_0} f_1(t) + a_m^s \sqrt{E_0} f_2(t), \quad m = 1, \ldots, M, \quad 0 \le t \le T$$

Hence

$$s_m(t) \leftrightarrow \mathbf{s_m} = \sqrt{E_0} \left(a_m^c, a_m^s \right)$$

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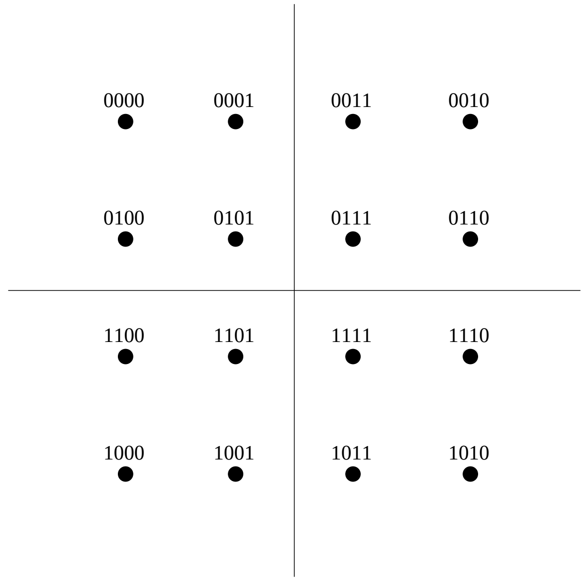
M-ary QAM

For the case when $M = 2^k$, k even, the resulting signal space diagram has a "square constellation." In this case the QAM signal can be thought of as 2 PAM signals in quarature. For $M = 2^k$, k odd, the constellation takes on a "cross" form. For example, 16-QAM constellation is



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M-ary PSK

Phase shift keyed (PSK) signals transmit information in the carrier phase. During any baud interval, the transmitted waveform is

$$s_m(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \theta_k\right) \quad , 0 \le t \le T$$

where

$$\theta_k \in \left\{ 2\pi \frac{(m-1)}{M}, \quad m = 1, \ldots, M \right\}$$

We can rewrite this in the form

$$s_m(t) = \sqrt{\frac{2E}{T}} \Bigg(\cos\theta_m \cos 2\pi f_c t - \sin\theta_m \sin 2\pi f_c t \Bigg) \quad , m = 1, \ldots, M$$

Using the same basis functions as QAM, we have

$$s_m(t) \leftrightarrow \mathbf{s_m} = \sqrt{E_0} \Bigg(\cos\theta_m, \sin\theta_m \Bigg)$$

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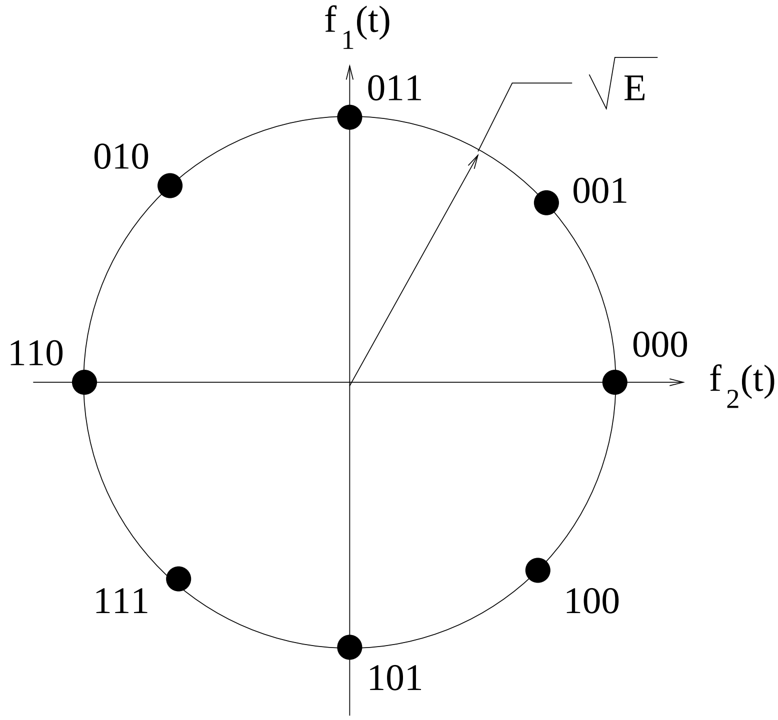
8-PSK Constellation



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M-ary FSK

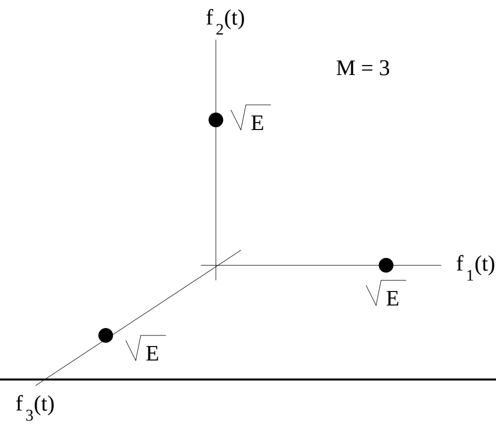
For Frequency shift keyed (FSK) signals, the transmitted signal during any given baud interval is

$$s_m(t) = A\cos(2\pi f_c t + 2\pi f_m t)\,,\ 0 \le t \le T$$

where

$$f_m = (m-1)\Delta_f, \quad m = 1, \ldots, M$$

We have seen before that the choice $\Delta_f = \frac{1}{2T}$ gives waveforms that are *orthogonal*.



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QAM Signals

Consider QAM signals defined on the interval $0 \leq t \leq T$:

$$s_m(t) = \sqrt{\frac{2E_0}{T}} \left(a_m^c \cos\left(2\pi f_c t\right) - a_m^s \sin\left(2\pi f_c t\right)\right) \qquad a_m^c, a_m^s \in \{\pm 1, \pm 3\}$$

The appropriate basis functions for the signal space are

$$f_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right) \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin\left(2\pi f_c t\right)$$

Then

$$\begin{aligned} s_m(t) &= \sqrt{E_0} a_m^c f_1(t) + \sqrt{E_0} a_m^s f_2(t) \\ \mathbf{s}_m &= \sqrt{E_0}(a_m^c, a_m^s) \end{aligned}$$

We randomly choose one of the 16 signals to transmit over an AWGN channel and receive $\mathbf{r} = \mathbf{s_m} + \mathbf{n}$, where $\mathbf{n} = (n_1, n_2)$, and the n_i are i.i.d. Gaussian random variables with variance $\sigma^2 = N_o/2$.

Our task is to find the probability of symbol error with minimum distance (or maximum likelihood) decisions.

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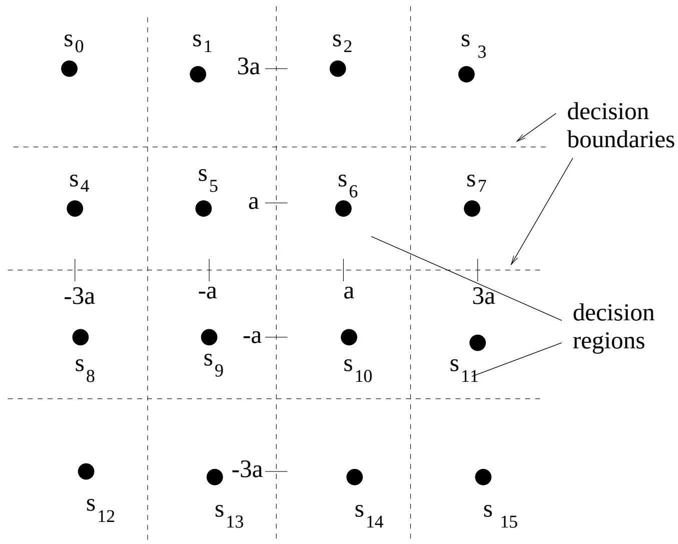
QAM Signals

To calculate the probability of symbol error, we first must define appropriate *decision regions* by placing *decision boundaries* between the signal points. For 16-QAM this is shown below.
Note that $a = \sqrt{E_0}$ **in the figure**.



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QAM Signals

For problems of this type, and especially for one or two-dimensional signal spaces (this problem is 2-D), it is often easier to calculate the probability of correct reception.

For this problem there are 3 cases to consider, since we can observe graphically that

$$\begin{aligned}
P_{C|\mathbf{s}_5} &= P_{C|\mathbf{s}_6} = P_{C|\mathbf{s}_9} = P_{C|\mathbf{s}_{10}} \\
P_{C|\mathbf{s}_0} &= P_{C|\mathbf{s}_3} = P_{C|\mathbf{s}_{12}} = P_{C|\mathbf{s}_{15}} \\
P_{C|\mathbf{s}_1} &= P_{C|\mathbf{s}_2} = P_{C|\mathbf{s}_4} = P_{C|\mathbf{s}_7} = P_{C|\mathbf{s}_8} = P_{C|\mathbf{s}_{11}} = P_{C|\mathbf{s}_{13}} = P_{C|\mathbf{s}_{14}}
\end{aligned}$$

All these quantities can be expressed in terms of the parameter

$$Q \equiv Q\left(\frac{\sqrt{E_0}}{\sigma}\right) \qquad \sigma^2 = \frac{N_o}{2}$$

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QAM Signals

All these quantities can be expressed in terms of the parameter

$$Q \equiv Q\left(\frac{\sqrt{E_0}}{\sigma}\right) \qquad \sigma^2 = \frac{N_o}{2}$$

We have

$$\begin{aligned} P_{C|\mathbf{s}_5} &= (1-2Q)^2 = 1 - 4Q + 4Q^2 \\ P_{C|\mathbf{s}_0} &= (1-Q)^2 = 1 - 2Q + Q^2 \\ P_{C|\mathbf{s}_1} &= (1-Q)(1-2Q) = 1 - 3Q + 2Q^2 \end{aligned}$$

Then

$$\begin{aligned} P_C &= \frac{1}{4}P_{C|\mathbf{s}_5} + \frac{1}{4}P_{C|\mathbf{s}_0} + \frac{1}{2}P_{C|\mathbf{s}_1} \\ &= 1 - 3Q + \frac{9}{4}Q^2 \end{aligned}$$

Finally, the probability of error is $P_e = 1 - P_C = 3Q - \frac{9}{4}Q^2$

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QAM Signals

Next, we need to find the *average* symbol energy. Remember that the energy in a symbol is equal to squared length of the signal vector.

In this case,

$$E_{\rm av} = \frac{1}{4}(E_0 + E_0) + \frac{1}{4}(9E_0 + 9E_0) + \frac{1}{2}(E_0 + 9E_0) = 10E_0$$

Hence, $E_0 = E_{\rm av}/10$, and

$$Q = Q\left(\frac{\sqrt{E_0}}{\sigma}\right) = Q\left(\sqrt{\frac{2E_0}{N_o}}\right) = Q\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right)$$

Finally,

$$P_e = 3Q\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right)$$

where

$$\frac{E_{\rm av}}{N_o} = \text{average symbol energy-to-noise ratio}$$

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QAM Signals

What about the bit error probability? That depends on the mapping of bits to symbols.

With Gray coding, a symbol error will usually result in one bit error. Certainly at most 4 bits errors will occur. Hence,

$$\frac{P_e}{4} \lessapprox P_b < P_e$$

Also, there are 4 bits per modulated symbol so that the average bit energy-to-noise ratio is

$$E_{b\ \mathrm{av}} = E_{\mathrm{av}}/4$$

So we can write

$$P_b \gtrapprox \frac{3}{4} Q\left(\sqrt{\frac{4}{5}\frac{E_{b\ \mathrm{av}}}{N_o}}\right) - \frac{9}{16} Q^2\left(\sqrt{\frac{4}{5}\frac{E_{b\ \mathrm{av}}}{N_o}}\right)$$

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Binary Error Probability

Consider two signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$.

The received signal vector is

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n}$$

A coherent maximum likelihood or minimum distance receiver decides in favor of the signal point $\mathbf{s}_1$ or $\mathbf{s}_2$ that is closest in Euclidean distance to the received signal point $\mathbf{r}$.

The error probability between $\mathbf{s}_1$ and $\mathbf{s}_2$ is

$$P(\mathbf{s}_1, \mathbf{s}_2) = Q\left(\sqrt{\frac{d_{12}^2}{2N_o}}\right)$$

where $d_{12}^2 = \|\mathbf{s}_1 - \mathbf{s}_2\|^2$ is the squared Euclidean distance between $\mathbf{s}_1$ and $\mathbf{s}_2$.

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Error Probability and Euclidean Distance

The error probability depends on the *Euclidean distance* between the signal vectors.

If we have two signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, separated by Euclidean distance $d_{12} = \|\mathbf{s}_1 - \mathbf{s_2}\|$, then the error probability is

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_o}}\right)$$

For BPSK $d_{12} = 2\sqrt{E}$
For BFSK $d_{12} = \sqrt{2E}$

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Voronoi Regions

Now suppose that we have a collection of M signal vectors, $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$.
The maximum likelihood receiver observes the received vector $\mathbf{r}$ and decides in favour of the signal vector that is closest in Euclidean distance (or squared Euclidean distance) to $\mathbf{r}$. That is

$$\hat{\mathbf{s}} = \text{argmin}_{\mathbf{s}_i} \|\mathbf{r} - \mathbf{s}_i\|^2$$

The received signal vector lies in the N-dimensional Euclidean space $\mathbf{R}^N$. Suppose that we form M partitions of $\mathbf{R}^N$ in the following fashion

$$R_i = \{\mathbf{r} : \|\mathbf{r} - \mathbf{s}_i\| = \min_j \|\mathbf{r} - \mathbf{s}_j\|\}$$

The $R_i, i = 1, \ldots, M$ are called *Voronoi* regions.

The maximum likelihood decision can be put in the form

$$\hat{\mathbf{s}} = \mathbf{s}_i \text{ whenever } \mathbf{r} \in R_i$$

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Error Probability

Under the assumption of equally likely transmitted symbols, the symbol error probability can be written as

$$P_M = 1 - P_C = 1 - \frac{1}{M}\sum_{j=1}^{M} P_{C|\mathbf{s_j}}$$

where $P_{C|\mathbf{s_j}}$ is the probability of a correct decision when $\mathbf{s}_j$ is sent.

The computation of P_M requires the set of probabilities $\{P_{C|\mathbf{s_j}}\}_{j=1}^{M}$.

However, a correct decision on $\mathbf{s}_j$ occurs if and only if the noise vector $\mathbf{n}$ does not move the received vector $\mathbf{r} = \mathbf{s}_j + \mathbf{n}$ outside the Voronoi region R_j, i.e.,

$$P_{C|\mathbf{s_j}} = P\{\mathbf{r} \in R_j\}$$

Using the conditional density function $p(\mathbf{r}|\mathbf{s}_j)$, we have

$$P_{C|\mathbf{s_j}} = \int_{R_j} \frac{1}{(\pi N_o)^{N/2}} e^{-\|\mathbf{r}-\mathbf{s}_j\|^2/N_o}$$

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Union Bound

In general, the Voronoi regions are very hard to determine so the integral

$$P_{C|\mathbf{s_j}} = \int_{R_j} \frac{1}{(\pi N_o)^{N/2}} e^{-\|\mathbf{r}-\mathbf{s}_j\|^2/N_o}$$

is very difficult if not impossible to compute, since we need to determine the upper and lower limits on an N-fold integral for a often complicated convex region in an N-dimensional space. In this case, upper and lower bounding techniques are useful.

Suppose we wish to compute $P_{C|\mathbf{s}_k}$.

Consider *only the pair* of signals $\mathbf{s}_k$ and $\mathbf{s}_j$. Let $\mathbf{s}_k$ be sent and let E_j denote the event that the receiver choose $\mathbf{s}_j$, hence making an error. Note that

$$P(E_j) = P(\mathbf{s}_k, \mathbf{s}_j)$$

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Union Bound

The probability of symbol error for $\mathbf{s}_k$ is

$$P_{E|\mathbf{s}_k} = P\left(\bigcup_{j \neq k} E_j\right)$$

The *union bound* on $P_{E|\mathbf{s}_k}$ is

$$P\left(\bigcup_{j \neq k} E_j\right) \leq \sum_{j \neq k} P(E_j)$$

Hence,

$$P_{E|\mathbf{s}_k} \leq \sum_{j \neq k} P(\mathbf{s}_k, \mathbf{s}_j)$$

If the $\mathbf{s}_i$ are equally likely, then

$$P_M = \frac{1}{M} \sum_{k=1}^{M} P_{E|\mathbf{s}_k} \leq \frac{1}{M} \sum_{k=1}^{M} \sum_{j \neq k} P(\mathbf{s}_k, \mathbf{s}_j)$$

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Union Bound

We have seen earlier that

$$P(\mathbf{s}_k, \mathbf{s}_j) = Q\left(\sqrt{\frac{d_{kj}^2}{2N_o}}\right)$$

where $d_{kj}^2 = \|\mathbf{s}_k - \mathbf{s}_j\|^2$.

Note that $Q(x)$ decreases with x. Hence, a further upper bound can be obtained by using the minimum distance $d_{\min} = \min_{j,k} d_{kj}$ and noting that

$$P(\mathbf{s}_k, \mathbf{s}_j) \;=\; Q\left(\sqrt{\frac{d_{kj}^2}{2N_o}}\right) \leq Q\left(\sqrt{\frac{d_{\min}^2}{2N_o}}\right)$$

Hence,

$$P_M \leq (M-1) Q\left(\sqrt{\frac{d_{\min}^2}{2N_o}}\right)$$

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Signal Set Rotation

The probability of symbol error is invariant to any rotation of the signal constellation $\{\mathbf{s}_i\}_{i=1}^{M}$ about the origin of the signal space. This is a consequence of two properties.

First, the probability of symbol error depends solely on the set of Euclidean distances $\{d_{jk}\}, j \neq k$ between the signal vectors in the signal constellation.

Second, the AWGN is circularly symmetric in all directions of the signal space.

A signal constellation can be rotated about the origin of the signal space, by multiplying each N-dimensional signal vector by an $N \times N$ unitary matrix $\mathbf{Q}$. A unitary matrix has the property $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = I$, where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$, and I is the $N \times N$ identity matrix.
The rotated signal vectors are equal to

$$\hat{\mathbf{s}}_i = \mathbf{s}_i\mathbf{Q}, \quad i = 1, \ldots, M \ .$$

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Signal Set Rotation

Correspondingly, the noise vector $\mathbf{n}$ is replaced with its rotated version

$$\hat{\mathbf{n}} = \mathbf{n}\mathbf{Q} \ .$$

The rotated noise vector $\hat{\mathbf{n}}$ is a vector of complex Gaussian random variables that is completely described by its mean and covariance matrix. The mean is

$$\mathrm{E}[\hat{\mathbf{n}}] = \mathrm{E}[\mathbf{n}]\mathbf{Q} = \mathbf{0} \ .$$

The covariance matrix is

$$\begin{aligned}
\mathbf{\Lambda}_{\hat{\mathbf{n}}\hat{\mathbf{n}}} &= \mathrm{E}[\hat{\mathbf{n}}^T\hat{\mathbf{n}}] \\
&= \mathrm{E}[(\mathbf{n}\mathbf{Q})^T\mathbf{n}\mathbf{Q}] \\
&= \mathrm{E}[\mathbf{Q}^T\mathbf{n}^T\mathbf{n}\mathbf{Q}] \\
&= \mathbf{Q}^T\mathrm{E}[\mathbf{n}^T\mathbf{n}]\mathbf{Q} \\
&= \frac{N_o}{2}\mathbf{Q}^T\mathbf{Q} = \frac{N_o}{2}\mathbf{I} \ .
\end{aligned}$$

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Signal Set Translation

Next consider a translation of the signal set such that

$$\hat{\mathbf{s}}_i = \mathbf{s}_i - \mathbf{a}, \quad i = 1, \ldots, M \ ,$$

where $\mathbf{a}$ is a constant vector. In this case, the error probability remains the same since $\hat{d}_{jk} = \tilde{d}_{jk}, j \neq k$. However, the average energy in the signal constellation is altered by the translation and becomes

$$\begin{aligned}
\hat{E}_{\text{av}} &= \sum_{i=1}^{M} \|\hat{\mathbf{s}}_i\|^2 P_i \\
&= \sum_{i=1}^{M} \|\mathbf{s}_i - \mathbf{a}\|^2 P_i \\
&= \sum_{i=1}^{M} \left\{ \|\mathbf{s}_i\|^2 - 2\mathbf{s}_i \cdot \mathbf{a} + \|\mathbf{a}\|^2 \right\} P_i \\
&= \sum_{i=1}^{M} \|\mathbf{s}_i\|^2 P_i - 2\left(\sum_{i=1}^{M} \mathbf{s}_i P_i\right) \cdot \mathbf{a} + \|\mathbf{a}\|^2 \sum_{i=1}^{M} P_i \\
&= E_{\text{av}} - 2\left\{ \mathrm{E}[\ \mathbf{s}\ ] \cdot \mathbf{a} \right\} + \|\mathbf{a}\|^2 \qquad (1)
\end{aligned}$$

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Signal Set Translation

where E_{av} is the average energy of the original signal constellation and $\mathrm{E}[\ \mathbf{s}\ ] = \sum_{i=0}^{M-1} \mathbf{s}_i P_i$ is its centroid (or center of mass).

Differentiating (1) with respect to the vector $\mathbf{a}$ and setting the result equal to zero will yield the translation that minimizes the average energy in the translated signal constellation. This gives

$$\mathbf{a}_{\text{opt}} = \mathrm{E}[\ \tilde{\mathbf{s}}\ ]\ \ .$$

Note that the center of mass of the translated signal constellation is at the origin, and the minimum average energy in the translated signal constellation is

$$\hat{E}_{\min} = E_{\text{av}} - \|\mathbf{a}_{\text{opt}}\|^2\ \ .$$

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