# EE4061 <br> Communication Systems 

Week 12<br>Intersymbol Interference<br>Nyquist Pulse Shaping

[^0]
## Intersymbol Interference (ISI)



An ideal channel $c(t)$ only scales and time shifts the signal $g(t)$, but otherwise leaves it undistorted, i.e. for an ideal channel

$$
\begin{aligned}
c(t) & =\alpha \delta\left(t-t_{o}\right) \\
g(t) * c(t) & =\alpha g\left(t-t_{o}\right) \\
C(f) & =\alpha e^{-j 2 \pi f t_{o}}
\end{aligned}
$$

[^1]

Amplitude and phase response for an ideal channel.
For a more general, non-ideal, channel, let

$$
\begin{aligned}
p(t) & =g(t) * c(t) * h(t) \\
& \imath \\
P(f) & =G(f) C(f) H(f)
\end{aligned}
$$

Then $y(t)=\sum_{n} a_{n} p(t-n T)+n(t)$, where $n(t)=w(t) * h(t)$

[^2]\[

$$
\begin{aligned}
y_{i}=y(i T) & =\sum_{n} a_{n} p((i-n) T)+n(i T) \\
& =\sum_{n} a_{n} p_{i-n}+n_{i}
\end{aligned}
$$
\]

where $p_{i-n}=p((i-n) T), n_{i}=n T$

$$
y_{i}=a_{i} p_{0}+\sum_{n \neq i} a_{n} p_{i-n}+n_{i}
$$

$a_{i} p_{o}-$ desired term,
$\sum_{n \neq i} a_{n} p_{i-n}-\mathrm{ISI}$
$n_{i}-$ noise
In the absence of ISI and noise, $y_{i}=a_{i} p_{0}$. Any pulse $p(t)$, such that the sampled pulse satisfies the condition

$$
p_{i}=p(i T)=\left\{\begin{array}{cc}
p_{0} & i=0 \\
0 & i \neq 0
\end{array}=p_{0} \delta_{i 0}\right.
$$

yields zero ISI

[^3]
## Bandlimited Pulse Shaping

What overall pulse shapes $p(t), p(t)=g(t) * c(t) * h(t)$, will yield zero ISI?
Suppose $P(f)=\frac{1}{2 W} \operatorname{rect}\left(\frac{f}{2 W}\right)$, where $W=1 / 2 T=R / 2, T$ is the baud duration, $R$ is the baud rate


$p(t)=\operatorname{sinc}(2 W t)=\operatorname{sinc}(t / T), T=1 / 2 W$.
Note that

$$
\begin{aligned}
p_{i}=p(i T) & = \begin{cases}1 & i=0 \\
0 & i \neq 0\end{cases} \\
& =\delta_{i 0}
\end{aligned}
$$

This pulse results in zero ISI. Note that $p(t)$ is noncausal.

[^4]
## ISI - Problems



## Problems:

1. $P(f)=\frac{1}{2 W} \operatorname{rect}\left(\frac{f}{2 W}\right)$ is and ideal low pass filter that is not realizable.
2. $p(t)$ decays slowly with time. It decreases with $1 /|t|$ for large $t$. Therefore, it is very sensitive to sampler phase, i.e., a small error in the sampler timing phase can lead to significant ISI.

We desire a pulse $p(t)$ that is realizable and has tails that decay quickly in time.

[^5]
## Problems with 'sinc' pulse

$$
\begin{aligned}
y(t) & =\sum_{n} a_{n} p(t-n T) \\
y(\Delta t) & =\sum_{n} a_{n} p(\Delta t-n T) \\
& =\sum_{n} a_{n} \frac{\sin [\pi(\Delta t-n T) / T]}{\pi(\Delta t-n T) / T} \\
& =\sum_{n} \frac{\sin (\pi \Delta t / T) \cos (\pi n)-\cos (\pi \Delta t / T) \sin (\pi n)}{\pi \Delta t / T-n \pi} \\
& =\sum_{n} a_{n} \frac{(-1)^{n} \sin (\pi \Delta t / T)}{\pi \Delta t / T-n \pi} \\
& =a_{0} \operatorname{sinc}(\Delta t / T)+\frac{\sin (\pi \Delta t / T)}{\pi} \sum_{n \neq 0} \frac{a_{n}(-1)^{n}}{\Delta t / T-n}
\end{aligned}
$$

Last term is not absolutely summable.

We have seen $y_{i}=y(i T)=a_{i} p_{0}+\sum_{n \neq i} a_{n} p_{i-n}+n_{i}$
where $p_{k}=p(k T), n_{i}=n(i T)$.

[^6]
## Matched Filtering and Pulse Shaping



- To maximize the signal-to-noise ratio at the output of the receiver filter $h(t)$, in theory we match the receiver filter to the received pulse $\hat{g}(t)=g(t) * c(t)$, i.e., $h(t)=\tilde{g}(T-t)$. However, if $c(t)$ is unknown, then so is $h(t)$.
- Practical Solution: Choose $h(t)$ matched to the transmitted pulse $g(t)$, i.e., choose $h(t)=g(T-t)$, over-sample by a factor of 2 , and process 2 samples per baud interval.
- This is optimal, similar to the case when $c(t)$ is known, but the proof is beyond the scope of this course.

[^7]
## Matched Filtering and Pulse Shaping

- To design the transmit and receiver filters, we will assume an ideal channel $c(t)=\delta(t)$, so that the overall pulse (ignoring time delay) is

$$
\begin{aligned}
p(t) & =g(t) * h(t) \\
& =g(t) * g(-t)
\end{aligned}
$$

- Taking the Fourier transform of both sides

$$
P(f)=G(f) G^{*}(f)=|G(f)|^{2}
$$

- Hence

$$
|G(f)|=\sqrt{|P(f)|}
$$

- For many practical pulses, $g(t)$, we will also see that $g(t)=g(-t)$, i.e., the pulse is even in $t$, so that $h(t)=g(t)$. This means that the transmit and receive matched filters are identical filters.

[^8]
## Conditions for ISI free transmission

The condition for ISI-free transmission is

$$
p_{k}=\delta_{k 0} p_{0}=\left\{\begin{array}{cc}
p_{0} & k=0 \\
0 & k \neq 0
\end{array}\right.
$$

That is, $p(t)$ must have equally spaced zero crossings, separated by $T$ seconds. Theorem: The pulse $p(t)$ satisfies $p_{k}=\delta_{k 0} p_{0}$ iff

$$
P_{\sum}(f) \triangleq \frac{1}{T} \sum_{n=-\infty}^{\infty} P(f+n / T)=p_{0}
$$

That is the folded spectrum $P_{\sum}(f)$ is flat.




[^9]
## ISI free transmission

## Proof:

$$
\begin{align*}
p_{k} & =\int_{-\infty}^{\infty} P(f) e^{j 2 \pi f k T} d f \\
& =\sum_{n=-\infty}^{\infty} \int_{(2 n-1) / 2 T}^{(2 n+1) / 2 T} P(f) e^{j 2 \pi f k T} d f \quad f^{\prime}=f-n / T \\
& =\sum_{n=-\infty}^{\infty} \int_{-1 / 2 T}^{1 / 2 T} P\left(f^{\prime}+n / T\right) e^{j 2 \pi k\left(f^{\prime}+n / T\right) T} d f^{\prime} \\
& =\int_{-1 / 2 T}^{1 / 2 T} e^{j 2 \pi f^{\prime} k T}\left[\sum_{n=-\infty}^{n=\infty} P\left(f^{\prime}+n / T\right)\right] d f^{\prime} \tag{1}
\end{align*}
$$

To prove sufficiency, we assume that $\sum_{n=-\infty}^{\infty} P\left(f^{\prime}+n / T\right)=p_{0} T$ is true. Then,

$$
p_{k}=p_{0} T \int_{-1 / 2 T}^{1 / 2 T} e^{j 2 \pi f^{\prime} k T} d f^{\prime}=\frac{\sin \pi k}{\pi k} p_{0}=\delta_{k 0} p_{k 0}
$$

To prove necessity, we have from (1)

$$
p_{k}=T \int_{-1 / 2 T}^{1 / 2 T} P_{\Sigma}\left(f^{\prime}\right) e^{j 2 \pi f^{\prime} k T} d f^{\prime}
$$

[^10]
## Nyquist Pulse

Hence, $p_{k}$ and $P_{\Sigma}(f)$ are a Fourier series pair, i.e.,

$$
P_{\Sigma}(f)=\sum_{k=-\infty}^{\infty} p_{k} e^{-j 2 \pi f k T}
$$

If $p_{k}=p_{0} \delta_{k 0}$ is assumed true, then from the above equation $P_{\Sigma}(f)=p_{0}$.

- Nyquist Pulse Shaping: A pulse $p(t)$ that yields zero-ISI is one having a folded spectrum that is flat.
- The pulse $p(t)$ can be generated by choosing $P(f)$ as shown on the following slide.

[^11]
## Nyquist Pulse Shaping





Note $P(f)=P_{N}(f)+P_{\text {od }}(f)$.
$P_{\text {od }}(f)$ can be any function that has skew symmetry about $f=W=1 / 2 T$.

[^12]
## Nyquist Pulse

Note that $P_{\Sigma}(f)$ is flat under this condition.


Example: Raised Cosine

$$
2 W P_{\text {od }}(f)= \begin{cases}-\frac{1}{2}-\frac{1}{2} \sin \frac{\pi(|f|-W)}{2 f_{x}} & W-f_{x} \leq|f| \leq W \\ \frac{1}{2}-\frac{1}{2} \sin \frac{\pi(|f|-W)}{2 f_{x}} & W \leq|f| \leq W+f_{x}\end{cases}
$$

$f_{x}=$ bandwidth expansion, $\frac{f_{x}}{W} \times 100=$ excess bandwidth (\%), $\alpha=\frac{f_{x}}{W}=$ roll off factor

$$
2 W P(f)= \begin{cases}1 & 0 \leq|f| \leq W-f_{x} \\ \frac{1}{2}\left[1-\sin \frac{\pi(|f|-W)}{2 f_{x}}\right] & W-f_{x} \leq|f| \leq W+f_{x} \\ 0 & |f| \geq W+f_{x}\end{cases}
$$

[^13]
## Raised Cosine Pulse



[^14]
## Raised Cosine Impulse Response

Impulse response - Since $P(f)$ is even, the inverse cosine transform yields

$$
\begin{aligned}
p(t) & =2 \int_{0}^{W+f_{x}} P(f) \cos 2 \pi f t d f \\
& =2 \cdot \frac{1}{2 W} \int_{0}^{W-f_{x}} \cos 2 \pi f t d f+2 \cdot \frac{1}{2 W} \int_{W-f_{x}}^{W+f_{x}} \frac{1}{2}\left[1-\sin \frac{\pi|f|-W}{2 f_{x}}\right] \cos 2 \pi f t d f \\
& =\frac{\sin 2 \pi W t}{2 \pi W t} \cdot \frac{\cos 2 \pi f_{x} t}{1-\left(4 f_{x} t\right)^{2}}
\end{aligned}
$$



[^15]
## Square Root Raised Cosine Pulse

- To implement a matched filter, we split the overall pulse $P(f)$ between the transmit and receive filters, i.e., $p(t)=g(t) * g(-t)$.
- However, $P(f)=G(f) G^{*}(f)=|G(f)|^{2}$, so that $|G(f)|=\sqrt{P(f)}$.
- With square-root raised cosine pulse shaping

$$
\sqrt{2 W}|G(f)|= \begin{cases}1 & 0 \leq|f| \leq w W-f_{x} \\ \sqrt{\frac{1}{2}\left[1-\sin \frac{\pi(|f|-W)}{2 f_{x}}\right]} & W-f_{x} \leq|f| \leq W+f_{x} \\ 0 & |f| \geq W+f_{x}\end{cases}
$$

- The impulse response is

$$
g(t)= \begin{cases}1-\beta+4 \beta / \pi & , t=0 \\ (\beta / \sqrt{2})((1+2 / \pi) \sin (\pi / 4 \beta)+(1-2 / \pi) \cos (\pi / 4 \beta)) & , t= \pm T / 4 \beta \\ \frac{4 \beta(t / T) \cos ((1+\beta) \pi t / T)+\sin ((1-\beta) \pi t / T)}{\pi(t / T)\left(1-(4 \beta t / T)^{2}\right)} & , \text { elsewhere }\end{cases}
$$

where $\alpha=f_{x} / W$.

[^16]
## Square Root Raised Cosine Pulse



Raised cosine and root raised cosine pulses with roll-off factor $\alpha=0.5$. The pulses are truncated to length $6 T$ and time shifted by $3 T$ to yield causal pulses.

[^17]
## $M$-ary QAM

- Quadrature Amplitude Modulation (QAM), the transmitted waveform in each baud interval takes on one of the following $M$ waveforms

$$
s_{m}(t)=\sqrt{\frac{2 E_{0}}{T}} g(t)\left(a_{m}^{c} \cos \left(2 \pi f_{c} t\right)-a_{m}^{s} \sin \left(2 \pi f_{c} t\right)\right)
$$

where

$$
a_{m}^{\{c, s\}} \in\{ \pm 1, \pm 3, \pm 5, \pm(M-1)\}
$$

and $2 E_{0}$ is the energy of the signal with the lowest amplitude, i.e., when $a_{m}^{c}, a_{m}^{s}= \pm 1$.

- You have seen this before for the case $g(t)=u_{T}(t)$; however, practical systems will use the root-raised cosine pulse for $g(t)$. Note that we use the normalization,

$$
E_{g}=\int_{-\infty}^{\infty} g^{2}(t) d t=1 .
$$

[^18]
## Eye Diagram with Ideal Nyquist Pulse



Eye diagram when $P(f)$ is an ideal low pass filter.

[^19]
## Eye Diagram with Raised Cosine Pulse



Eye diagram when $P(f)$ is a raised cosine filter with $\beta=0.35$.

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