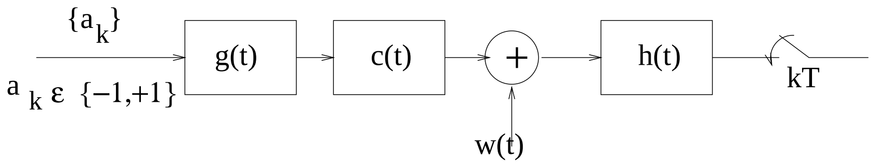
EE4601
Communication Systems

Week 13

Partial Response Signals

Objective

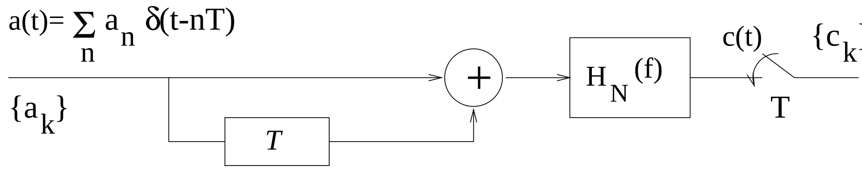
Objective: Signals with a baud rate of 2W symbols/sec in a bandwidth of W Hz with realizable filters.



$$
\begin{aligned}
\text{Assume} \quad c(t) &= \delta(t) \quad \text{(ideal channel)} \\
\text{Then} \quad h(t) &= g(T-t) \\
p(t) &= g(t) * g(-t) \\
P(f) &= |G(f)|^2
\end{aligned}
$$

Duobinary Signaling

Assume that $P(f)$ has the following form



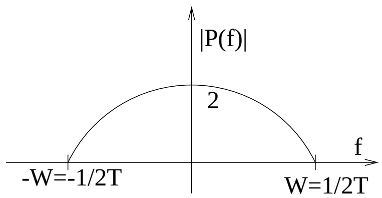
$$H_N(f) = \frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right) = \begin{cases} 1 & |f| < W \\ 0 & \text{else} \end{cases}$$

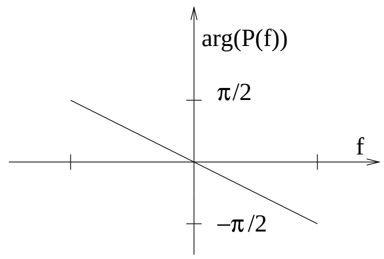
where $W = 1/2T$ i.e., the baud rate is $R = 1/T = 2W$ symbols/sec

$$\begin{aligned} P(f) &= (1 + e^{-j2\pi fT})H_N(f) \\ &= 2e^{-j\pi fT}\left(\frac{e^{j\pi fT} + e^{-j\pi fT}}{2}\right)H_N(f) \\ &= 2\cos(\pi fT)e^{-j\pi fT}H_N(f) \\ &= 2\cos(\pi fT)e^{-j\pi fT}\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right) \end{aligned}$$

Duobinary

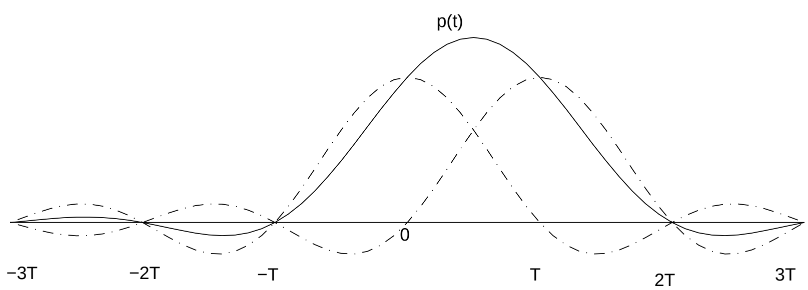
$$P(f) = \begin{cases} 2T\cos(\pi f T)e^{-j\pi f T} & |f| < W \\ 0 & \text{else} \end{cases}$$



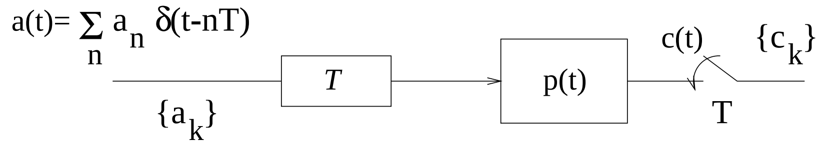


To get $p(t)$ we write

$$\left\{P(f) = H_N(f) + H_N(f)e^{j2\pi f T}\right\} \leftrightarrow \left\{p(t) = \text{sinc}\left(\frac{t}{T}\right) + \text{sinc}\left(\frac{t-T}{T}\right)\right\}$$



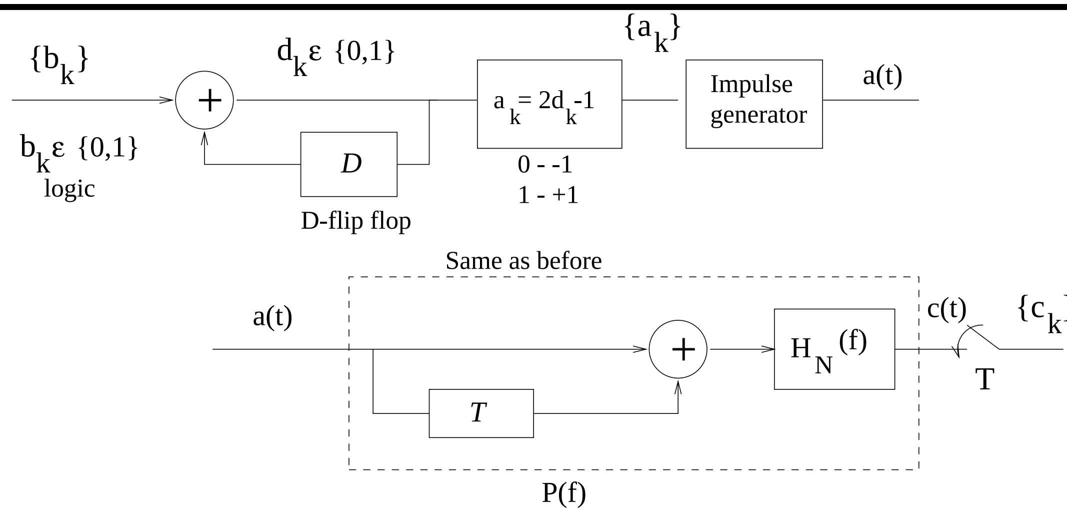
Duobinary



$$
\begin{aligned}
c(t) &= \sum_n a_n p(t - nT) \\
c_k = c(kT) &= \sum_n a_n p((k-n)T) = \sum_n a_n p_{k-n} \\
\text{But } \; p_j = p(jT) &= \begin{cases} 1 & j = 0, 1 \\ 0 & j \neq 0, 1 \end{cases} \\
\text{Therefore, } \; c_k &= a_k + a_{k-1}
\end{aligned}
$$

Since $a_k \in \{-1, +1\}$ $c_k \in \{-2, 0, 2\}$ (3-level)
We can recover $\{a_k\}$ from $\{c_k\}$ by $a_k = c_k - a_{k-1}$ assuming an initial value, e.g. $a_0 = -1$ or $+1$. This is called <u>decision feedback</u> detection.
Problem : Errors due to noise propagate, i.e., $\hat{a_k} = c_k - \widehat{a_{k-1}}$.
If $\widehat{a_{k-1}}$ is in error then $\hat{a_k}$ is likely to be in error.

Precoding



$$d_k = b_k \oplus d_{k-1} = b_k + d_{k-1} (\mathrm{mod}\ 2)$$

Example:

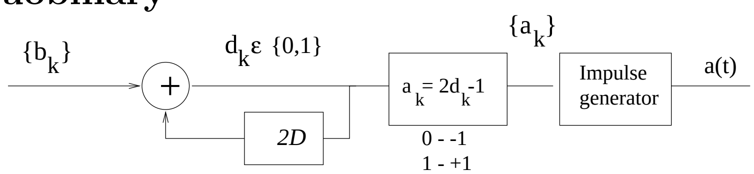
$\{b_k\}$		0	0	1	1	1	0	1	0	0	0
$\{d_k\}$	1	1	1	0	1	0	0	1	1	1	1
$\{a_k\}$	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$
$\{c_k\}$		$+2$	$+2$	0	0	0	-2	0	$+2$	$+2$	$+2$

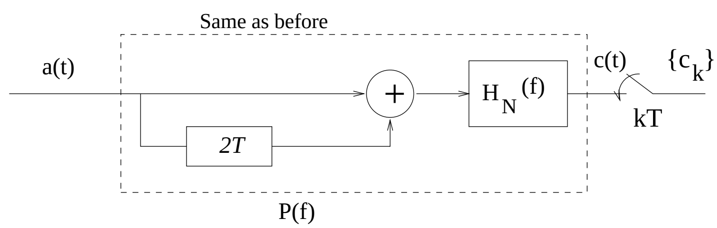
Modified Duobinary

Note that $c_k = \begin{cases} \pm 2 & \text{if} \quad b_k = 0 \\ 0 & \text{if} \quad b_k = 1 \end{cases}$

Therefore $\{b_k\}$ can be recovered from $\{a_k\}$ by using symbol by symbol detection.

Modified Duobinary

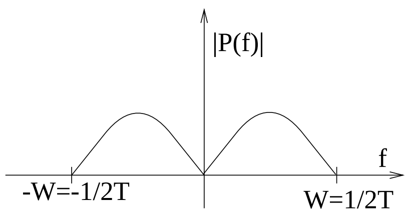


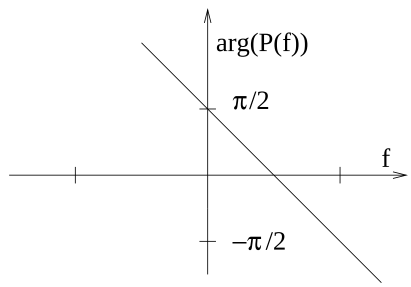


$a(t) = \sum_n a_n \delta(t - nT)$, $d_k = b_k \oplus d_{k-2}$, $c_k = a_k - a_{k-2}$.

Modified Duobinary

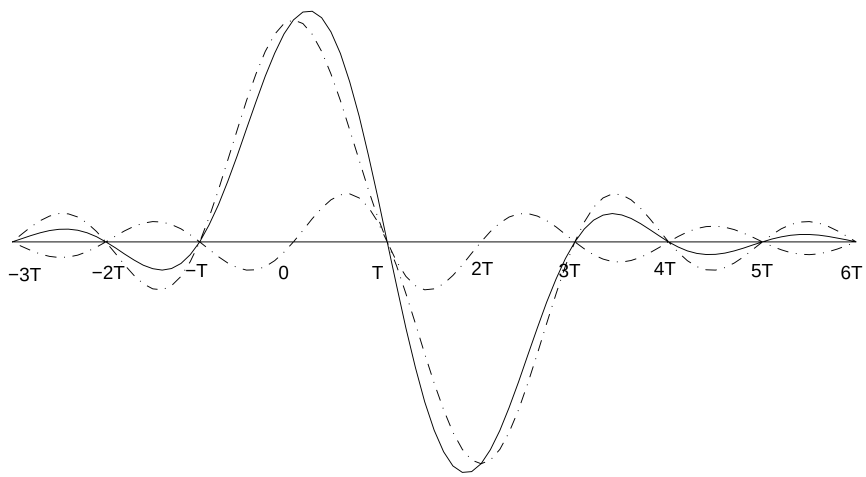
$$
\begin{aligned}
P(f) &= (1 - e^{-j4\pi fT}) H_N(f) \\
&= j2e^{-j2\pi fT} \left(\frac{e^{j2\pi fT} - e^{-j2\pi fT}}{j2} \right) H_N(f) \\
&= j2H_N(f) \sin 2\pi fT e^{-j2\pi fT} \\
&= \begin{cases} 2T \sin 2\pi fT e^{j(\pi/2 - 2\pi fT)} & |f| < 1/2T \\ 0 & |f| > 1/2T \end{cases}
\end{aligned}
$$





Modified Duobinary Pulse

$$P(f) = H_N(f) - e^{-j4\pi fT} H_N(f)$$
$$p(t) = \mathrm{sinc}(t/T) - \mathrm{sinc}((t-2T)/T)$$



Modified Duobinary with Precoding

Example:

$\{b_k\}$			0	0	1	1	1	0	1	0	0	0
$\{d_k\}$	1	1	1	1	0	0	1	0	0	0	0	0
$\{a_k\}$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	-1	-1	-1	-1	-1
$\{c_k\}$			0	0	-2	-2	2	0	-2	0	0	0

Note: $c_k = \begin{cases} \pm 2 & \text{if } b_k = 1 \\ 0 & \text{if } b_k = 0 \end{cases}$

Duobinary Error Probability

Here we consider the error probability of precoded duobinary signaling with *symbol-by-symbol* detection.

The transmit and receiver filters are implemented as the root-duobinary pulse, such that

$$|G(f)| = |H(f)| = \sqrt{P(f)}$$

We note that the noise process at the output of the receiver filter has power spectral density

$$\Phi_{nn}(f) = \frac{N_o}{2}|H(f)|^2 = \frac{N_o}{2}P(f)$$

Since $p(t)$ is not a Nyquist pulse, i.e., $p_k = p(kT) \neq \delta_{k0}$, the noise samples are correlated, and, hence the symbol-by-symbol detector is *suboptimal.* This loss can be recovered by using a *sequence detector*, but we will not discuss here.

The Gaussian noise samples at the output of the receiver matched filter $H(f)$ are zero mean and have variance

$$\sigma_n^2 = \frac{N_o}{2}\int_{-\infty}^{\infty} P(f)df = \frac{N_o}{2}\int_{-1/2T}^{1/2T} 2\cos(\pi f T)df = \frac{2N_o}{\pi}$$

Duobinary Error Probability

Note that the sampled outputs of the matched filter have the Gaussian density function

$$y_k \sim \begin{cases} N(\pm 2, 2N_o/\pi) & , \ x_k = 0 \\ N(0, 2N_o/\pi) & , \ x_k = 1 \end{cases}$$

where the means ± 2 each occur with probability $1/4$ and the mean 0 occurs with probability $1/2$.

Assuming that the receiver makes decisions according to

$$\hat{b}_k = \begin{cases} 1, & |y_k| < 1 \\ 0, & |y_k| > 1 \end{cases}$$

we have the probability of error

$$P_b = \frac{1}{4} \cdot Q + \frac{1}{4} \cdot Q + \frac{1}{2} \cdot 2Q = \frac{3}{2} Q$$

where

$$Q = Q\left(\frac{1}{\sigma}\right) = Q\left(\sqrt{\frac{\pi}{2N_o}}\right)$$

Duobinary Error Probability

The energy per bit is

$$E_b = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} P(f) df = \frac{4}{\pi}$$

Hence, $\frac{\pi}{4} E_b = 1$ and

$$Q = Q\left(\sqrt{\frac{\pi}{2N_o} \cdot \frac{\pi}{4} E_b}\right) = Q\left(\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{2E_b}{N_o}}\right)$$

and

$$P_b = \frac{3}{2} Q\left(\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{2E_b}{N_o}}\right)$$

When compared to binary antipodal signaling with

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

the loss in E_b/N_o performance is $-10\log_{10}(\pi/4)^2 = 2.1$ dB.