EE4601
Communication Systems

Week 15

Noncoherent Detection

Differential Detection

Rayleigh Random Variable

Let:

$$R = \sqrt{X^2 + Y^2},\ \Phi = \mathrm{Tan}^{-1}\frac{Y}{X}$$

$$\text{where } X, Y \sim N(0, \sigma^2)$$

Then:

$$X = R\cos\Phi$$

$$Y = R\sin\Phi$$

$$f_{R,\Phi}(r, \phi) = f_{XY}(r\cos\phi, r\sin\phi)|J|$$

where

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2 + y^2}{2\sigma^2}\right\}$$

and

$$|J| = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r$$

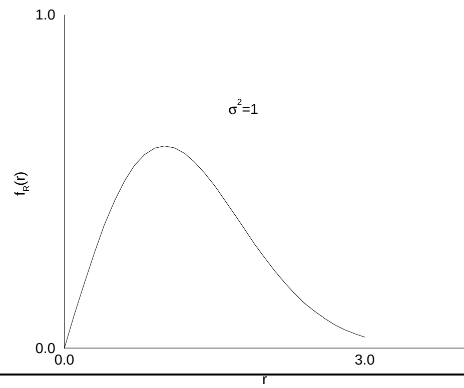
Rayleigh Random Variable

$$f_{R,\Phi}(r,\phi) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Phi}(r,\phi)d\phi = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} \quad , r \geq 0$$

$f_R(r)$ has a Rayleigh distribution and, for our problem,

$$f_{l_2}(x) = \frac{2x}{N_0}\exp\left\{-\frac{x^2}{N_0}\right\} \quad , x \geq 0$$



[0] ©2009, Georgia Institute of Technology (lecture 24_3)

Rician Random Variable

Let:

$$R = \sqrt{X^2 + Y^2}, \ \Phi = \mathrm{Tan}^{-1}\frac{Y}{X}$$

$$\text{where } X \sim N(\sqrt{E}\cos\Phi_1, \sigma^2), \ Y \sim N(\sqrt{E}\sin\Phi_1, \sigma^2)$$

Using the same procedure as before,

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x-\sqrt{E}\cos\phi_1)^2 + (y-\sqrt{E}\sin\phi_1)^2}{2\sigma^2}\right\} \\
f_{R,\Phi}(r,\phi) &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+E}{2\sigma^2}\right\}\exp\left\{\frac{\sqrt{E}r}{\sigma^2}\cos(\phi-\phi_1)\right\} \\
f_R(r) &= \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+E}{2\sigma^2}\right\} I_0\left(\frac{\sqrt{E}r}{\sigma^2}\right), \ r \geq 0,
\end{aligned}$$

where

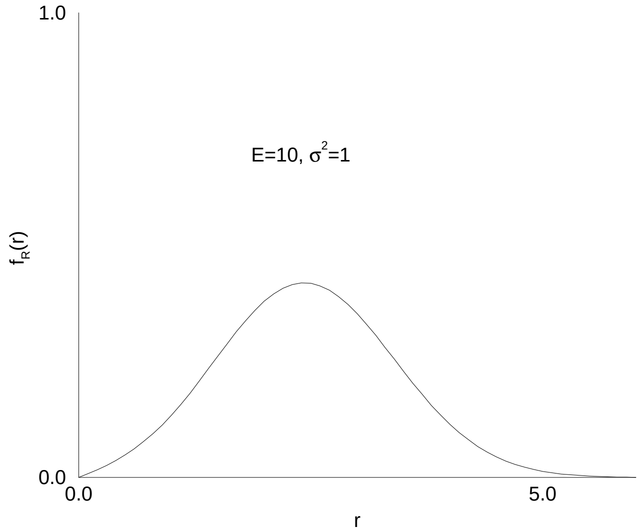
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp\{x\cos\phi\}\,d\phi$$

is a zero-order modified Bessel function of the first kind.

Rician Random Variable

For our problem

$$f_{l_1}(x) = \frac{2x}{N_o} \exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \quad , x \geq 0$$



Bit Error Probability

$$
\begin{aligned}
P_b &= P(l_2 > l_1 | s_1(t) \text{ sent}) \\
&= \int_0^\infty P(l_2 > l_1 | l_1 \text{ and } s_1(t) \text{ sent}) f(l_1) dl_1
\end{aligned}
$$

$$
\begin{aligned}
P(l_2 > l_1 | l_1 \text{ and } s_1(t) \text{ sent}) &= \int_{l_1}^\infty f(l_2) dl_2 \\
&= \exp\left\{ -\frac{l_1^2}{N_0} \right\}
\end{aligned}
$$

$$
\begin{aligned}
P_b &= \int_0^\infty \exp\left\{ -\frac{l_1^2}{N_0} \right\} \frac{2l_1}{N_0} \exp\left\{ -\frac{l_1^2 + E}{N_0} \right\} I_0\left(\frac{2l_1\sqrt{E}}{N_0} \right) dl_1 \\
&= \int_0^\infty \frac{2l_1}{N_0} \exp\left\{ -\frac{2l_1^2 + E}{N_0} \right\} I_0\left(\frac{2l_1\sqrt{E}}{N_0} \right) dl_1
\end{aligned}
$$

Error Probability

Define $v = \frac{2l_1}{\sqrt{N_0}}$. Then,

$$P_b = \frac{1}{2}\exp\left\{-\frac{E}{2N_0}\right\}\int_0^\infty v\exp\left\{-\frac{v^2+a^2}{2}\right\}I_0\left(av\right)dv$$

where $a = \sqrt{\frac{E}{N_0}}$.

However, the integral is a Rice pdf that is being integrated over its entire range, i.e., the integral is equal to 1.

Therefore,

$$P_b = \tfrac{1}{2}\exp\left\{-\tfrac{E}{2N_0}\right\}$$

ASK Signals

Noncoherent detection can be used for ASK signals as well. Consider the two signals

$$
\begin{aligned}
s_1(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_c t),\ 0 \le t \le T \\
s_2(t) &= 0,\ 0 \le t \le T
\end{aligned}
$$

We use just a single energy detector at frequency f_c.

If $s_1(t)$ is sent, the pdf of the detector output, ℓ, is

$$
f_{\ell|s_1}(x) = \frac{2x}{N_0} \exp\left\{-\frac{x^2}{N_0}\right\} \quad , x \ge 0
$$

If $s_2(t)$ is sent, the pdf of the detector output is

$$
f_{\ell|s_2}(x) = \frac{2x}{N_o} \exp\left\{-\frac{x^2 + E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \quad , x \ge 0
$$

ASK Signals

The optimum decision threshold, λ, is where the two conditional pdfs cross. To find the threshold, we solve

$$
\begin{aligned}
f_{\ell|s_1}(x) &= f_{\ell|s_2}(x) \\
\frac{2x}{N_0}\exp\left\{-\frac{x^2}{N_0}\right\} &= \frac{2x}{N_o}\exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \\
1 &= \exp\left\{-\frac{E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right)
\end{aligned}
$$

The solution to the above equation gives λ_{opt}.

ASK Signals

Then

$$
\begin{aligned}
P_b &= \frac{1}{2}\int_{\lambda_{\text{opt}}}^{\infty} f_{\ell|s_1}(x)dx + \frac{1}{2}\int_{0}^{\lambda_{\text{opt}}} f_{\ell|s_2}(x)dx \\
&= \frac{1}{2}\exp\left\{-\frac{{\lambda_{\text{opt}}}^2}{N_0}\right\} + \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{\frac{2{\lambda_{\text{opt}}}^2}{N_o}}\right)\right]
\end{aligned}
$$

where $Q(x,y)$ is called a Marcum-Q function, and

$$
Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{\frac{2{\lambda_{\text{opt}}}^2}{N_o}}\right) = \int_{\lambda_{\text{opt}}}^{\infty} \frac{2x}{N_o}\exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right)dx
$$

Unfortunately, the result does not exist in closed form.

Differential Detection of PSK

With differential PSK (DPSK), information can be transmitted in the differential carrier phase between sucessive symbols.

DPSK can be detected noncoherently by using differentially coherent detection, where the receiver compares the phase of the received signal between two successive signaling intervals.

Suppose that binary DPSK is used. Let θ_n denote the absolute carrier phase for the nth symbol, and $\Delta\theta_n = \theta_n - \theta_{n-1}$ denote the differential carrier phase. Several mappings exist between the differential carrier phase and source symbols. Here we consider the mapping

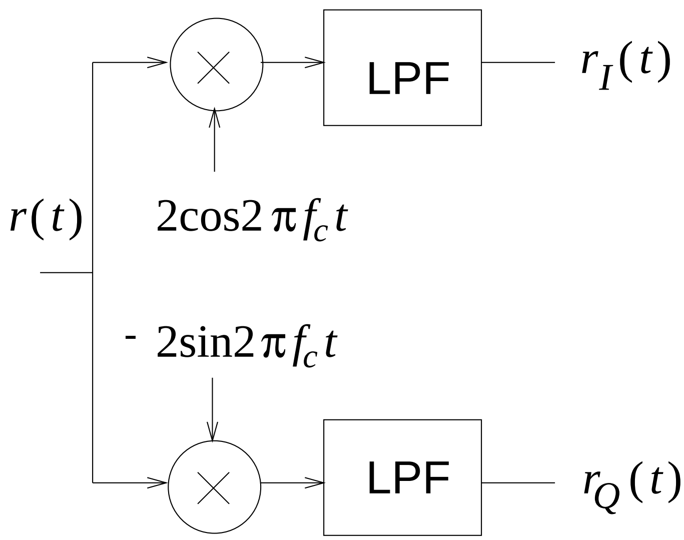
$$\Delta\theta_n = \begin{cases} 0 \quad , & x_n = +1 \\ \pi \quad , & x_n = -1 \end{cases}$$

The transmitted bandpass waveform is

$$s(t) = \sum_n g(t - nT)\cos(2\pi f_c t + \theta_n)$$

Quadrature Demodulator

Received bandpass signal is $r(t) = s(t) + n(t)$.



Receiver for DPSK

The received signal is

$$r(t) = \sum_n g(t - nT)\cos(2\pi f_c t + \theta_n + \phi) + n(t)$$

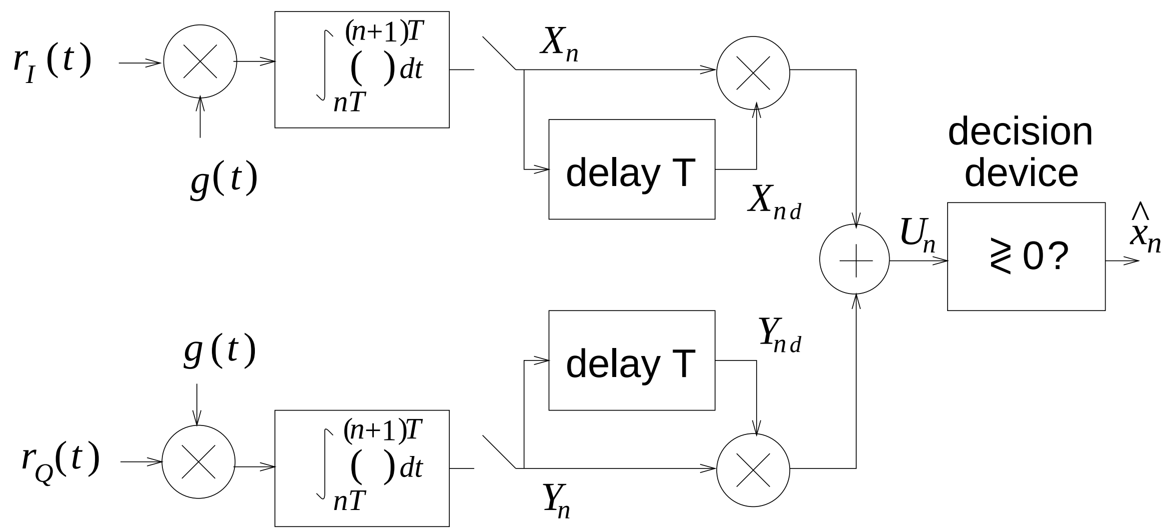
where

$$n(t) = n^I(t)\cos(2\pi f_c t) - n^Q(t)\sin(2\pi f_c t)$$

After quadrature demodulation we have

$$\begin{aligned}
r_I(t) &= [2r(t)\cos(2\pi f_c t)]_{\mathrm{LP}} \\
&= \sum_n g(t - nT)\cos(\theta_n + \phi) + n^I(t) \\
r_Q(t) &= [-2r(t)\sin(2\pi f_c t)]_{\mathrm{LP}} \\
&= \sum_n g(t - nT)\sin(\theta_n + \phi) + n^Q(t)
\end{aligned}$$

Receiver for DPSK



Receiver for DPSK

The values of X_k, X_{kd}, Y_k and Y_{kd} are

$$\begin{aligned} X_n &= 2E\cos(\theta_n + \phi) + n^I \\ X_{nd} &= 2E\cos(\theta_{n-1} + \phi) + n_d^I \\ Y_n &= 2E\sin(\theta_n + \phi) + n^Q \\ Y_{nd} &= 2E\sin(\theta_{n-1} + \phi) + n_d^Q \end{aligned}$$

where E is the bit energy given by

$$E = \frac{1}{2}\int_0^T g^2(t)dt$$

The noise terms are

$$\begin{aligned} n^I &= \int_{nT}^{(n+1)T} n^I(t)g(t)dt \ , \qquad n^Q = \int_{nT}^{(n+1)T} n^Q(t)g(t)dt \\ n_d^I &= \int_{(n-1)T}^{nT} n^I(t)g(t)dt \ , \qquad n_d^Q = \int_{(n-1)T}^{nT} n^Q(t)g(t)dt \end{aligned}$$

Note that n^I, n^Q, n_d^I and n_d^Q are all i.i.d $\sim N(0, 2EN_o)$.

Receiver for DPSK

In the absence of noise we have

$$\begin{aligned}
U_n &= X_n X_{nd} + Y_n Y_{nd} \\
&= 4E^2[\cos(\theta_n + \phi)\cos(\theta_{n-1} + \phi) + \sin(\theta_n + \phi)\sin(\theta_{n-1} + \phi)] \\
&= 4E^2 \cos(\theta_n - \theta_{n-1}) \\
&= 4E^2 \cos(\Delta\theta_n) \\
&= 4E^2 x_n
\end{aligned}$$

To evaluate the error probability, we need the density function of the detector output U_n given the transmitted symbol x_n in the presence of noise.

Note that U_n involves the sum of products of Gaussian random variables.

Density Functions

The conditional density function of U_n given x_n is [1]

$$f_{U_n|x_n}(u) = \begin{cases} \frac{1}{4EN_o} \exp\left\{\frac{x_n u - 2E^2}{2EN_o}\right\}, & -\infty < x_n u < 0 \\ \\ \frac{1}{4EN_o} \exp\left\{\frac{x_n u - 2E^2}{2EN_o}\right\} Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{\frac{2x_n u}{EN_o}}\right), & 0 < x_n u < \infty \end{cases}$$

where $Q(a,b)$ is the Marcum Q function, defined by

$$Q(a,b) = 1 - \int_0^b z e^{-\frac{z^2+a^2}{2}} I_0(za)\, dz$$

[1] G.L. Stüber, "Soft Decision Direct-Sequence DPSK Receivers," *IEEE Transactions on Vehicular Technology*, vol. 37, no. 3, pp. 151-157, August 1988.

Error Probability for DPSK

To obtain the bit error probability we assume that $x_n = \pm 1$ with equal probability.

Then suppose that $x_n = 1$ is transmitted so that the probability of error is the probability that U_n is less than zero.

We have

$$\begin{aligned} P_b &= \int_{-\infty}^{0} f_{U_n|x_n=1}(u)du \\ &= \int_{-\infty}^{0} \frac{1}{4EN_o} \exp\left\{\frac{u-2E^2}{2EN_o}\right\} du \\ &= \frac{1}{2}e^{-E/N_o} \end{aligned}$$

where $E = E_b$ is the bit energy.