EE4601
Communication Systems

Week 2

Review of Probability,
Important Distributions

Conditional Probability

Consider a sample space that consists of two events A and B.
The conditional probability $P(A|B)$ is the probability of the event A given that the event B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A and B are *independent* events, then

$$P(A \bigcap B) = P(A)P(B)$$

Hence,

$$P(A|B) = P(A)$$

and the occurrence of event B does not change the probability of occurrence of event A.

Disjoint vs. Independent Events

The probability of the union event "A or B" is

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$

If events A and B are *mutually exclusive* or *disjoint* then

$$P(A \bigcup B) = P(A) + P(B)$$

Note that mutually exclusive and independent events are two entirely different concepts. In fact, independent events A and B with non-zero probabilities, $P(A)$ and $P(B)$, cannot be mutually exclusive because $P(A \cap B) = P(A)P(B) > 0$. If they were mutually exclusive then we must have $P(A \cap B) = 0$.

Intuitively, if the events A and B are mutually exclusive, then the occurrence of the event A precludes the occurrence of the event B. Hence, the knowledge that A has occurred definitely affects the probability that B has occurred. So A and B are not independent.

Bayes' Theorem

Let events $A_i, i = 1, \ldots n$ be mutually exclusive such that $\cup_{i=1}^n A_i = S$, where S is the sample space. Let B be some event with non-zero probability. Then

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^n P(B|A_i)P(A_i)} \end{aligned}$$

where we use notation $P(A_i, B) = P(A_i \cap B)$.

For continuous random variables x and y with probability density functions $f(x)$ and $f(y)$, the conditional density $f(x|y)$ is

$$\begin{aligned} f(x|y) &= \frac{f(x, y)}{f(y)} \\ &= \frac{f(y|x)f(x)}{\int f(y|x)f(x)dx} \end{aligned}$$

Bayes' Theorem - Example

Suppose that a digital source generates 0's and 1's with unequal probabilities $Q(0) = q$ and $Q(1) = 1 - q$. The bits are transmitted over a binary symmetric channel (BSC), with inputs k and outputs j, and transmission probabilities $P(j|k)$ such that $P(1|0) = P(0|1) = p$ and $P(0|0) = P(1|1) = 1 - p$, where $P(j|k)$ is the probability that j is received given that k is transmitted.
If a "1" is received what is the probability that a "0" was transmitted?

$$\begin{aligned} P(k|j) &= \frac{P(j|k)Q(k)}{P(j|0)Q(0) + P(j|1)Q(1)} \\ P(0|1) &= \frac{P(1|0)Q(0)}{P(1|0)Q(0) + P(1|1)Q(1)} \\ &= \frac{pq}{pq + (1-p)(1-q)} \end{aligned}$$

What is the probability of bit error?

$$P_e = P(1|0)Q(0) + P(0|1)Q(1) = p$$

Random Variables

Consider the random variable X.

The *cumulative distribution function* (cdf) of X is

$$F_X(x) = P(X \leq x) \ , \quad 0 \leq F_X(x) \leq 1$$

The *complementary distribution function* (cdfc) of X is

$$F_X^c(x) = P(X > x) = 1 - F_X(x) \ , \quad 0 \leq F_X(x) \leq 1$$

The *probability density function* (pdf) of X is

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Bivariate Random Variables

Consider two random variables X and Y. The *joint* (cdf) of X and Y is

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) \ , \quad 0 \leq F_{XY}(x,y) \leq 1$$

The *joint* (cdfc) of X and Y is

$$F^c_{XY}(x,y) = P(X > x, Y > y) = 1 - F_{XY}(x,y) \ , \quad 0 \leq F_{XY}(x,y) \leq 1$$

The *joint* (pdf) of X and Y is

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \qquad F_{XY}(x) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)dxdy$$

The *marginal* pdfs of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \qquad f_Y(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

The *conditional* pdfs of X and Y are

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Statistical Averages

Consider any random variable X.

The mean of X is

$$\mu_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

The nth moment of X is

$$\mathrm{E}[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

The variance of X is

$$\begin{aligned} \sigma_X^2 &= \mathrm{E}[(X - \mu_X)^2] \\ &= \mathrm{E}[X^2 - 2X\mu_X + \mu_X^2] \\ &= \mathrm{E}[X^2] - 2\mathrm{E}[X]\mu_X + \mu_X^2 \\ &= \mathrm{E}[X^2] - \mu_X^2 \end{aligned}$$

Consider any function $g(X)$ of the random variable X. Then

$$\mathrm{E}[g^n(X)] = \int_{-\infty}^{\infty} g^n(x) f_X(x) dx$$

Joint Moments

Consider a pair of random variables X and Y. The joint moment of X and Y is

$$\mathrm{E}[X^i Y^j] = \int_{-\infty}^{\infty} x^i y^j f_{XY}(x,y) dx dy$$

The *covariance* of X and Y is

$$\begin{aligned} \mathrm{cov}[X,Y] &= \mathrm{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mu_Y - \mathrm{E}[Y]\mu_X + \mu_X\mu_Y \\ &= \mathrm{E}[XY] - \mu_X\mu_Y \end{aligned}$$

The *correlation coefficient* of X and Y is

$$\rho = \frac{\mathrm{cov}[X,Y]}{\sigma_X \sigma_Y}$$

Two random variables X and Y are *uncorrelated* iff $\mathrm{cov}[XY] = 0$.

Note that *independent* $\rightarrow$ *uncorrelated*.

Two random variables X and Y are *orthogonal* iff $\mathrm{E}[XY] = 0$.

Characteristic Functions

Consider the random variable X. The *characteristic* or *moment generating* function of X is

$$\Phi_X(v) = \mathrm{E}[e^{jvX}] = \int_{-\infty}^{\infty} f_X(x) e^{jvx} dx$$

Except for the sign of the exponent in the integrand, the characteristic function is just the Fourier transform of the pdf.
Taking the derivative of both sides n times and setting $v = 0$ gives

$$\left.\frac{d^n}{dv^n}\Phi_X(v)\right|_{v=0} = (j)^n \int_{-\infty}^{\infty} x^n f_X(x) dx$$

Recognizing the integral on the R.H.S. as the nth moment, we have

$$\left.(-j)^n \frac{d^n}{dv^n}\Phi_X(v)\right|_{v=0} = \mathrm{E}[x^n]$$

The pdf is inverse Fourier transform (note change in sign of exponent)

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(v) e^{-jvx} dv$$

Joint Characteristic Functions

Consider the random variables X and Y. The joint characteristic function is

$$\Phi_{XY}(v_1, v_2) = \mathrm{E}[e^{jv_1X+jv_2Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)e^{jv_1x+jv_2y}dxdy$$

If X and Y are independent, then

$$\begin{aligned}\Phi_{XY}(v_1, v_2) &= \mathrm{E}[e^{jv_1X+jv_2Y}] \\ &= \int_{-\infty}^{\infty} f_X(x)e^{jv_1x}dx \int_{-\infty}^{\infty} f_Y(y)e^{jv_2y}dy \\ &= \Phi_X(v_1)\Phi_Y(v_2)\end{aligned}$$

Moments can be generated according to

$$\mathrm{E}[XY] = -\frac{\partial^2\Phi_{XY}(v_1, v_2)}{\partial v_1 \partial v_2}\Big|_{v_1=v_2=0}$$

with higher order moments generated in a straight forward extension.

Binomial Distribution

Let X be a **Bernoulli random variable** such that $X = 0$ with probability $1-p$ and $X = 1$ with probability p. Although X is a discrete random random variable with an associated **probability distribution function**, it is possible to treat X as a continuous random variable with a **probability density function** (pdf) by using dirac delta functions. The pdf of X can be written as

$$p_X(x) = (1-p)\delta(x) + p\delta(x-1)$$

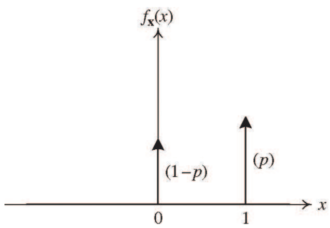
Let $Y = \sum_{i=1}^{n} X_i$, where the X_i are independent and identically distributed (iid) Bernoulli random variables. Then the random variable Y is an integer from the set $\{0, 1, \ldots, n\}$ and Y has the binomial probability distribution function

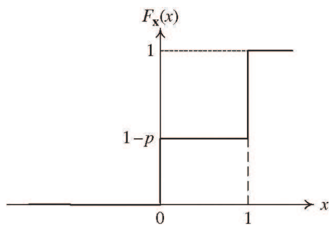
$$p_Y(k) = P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

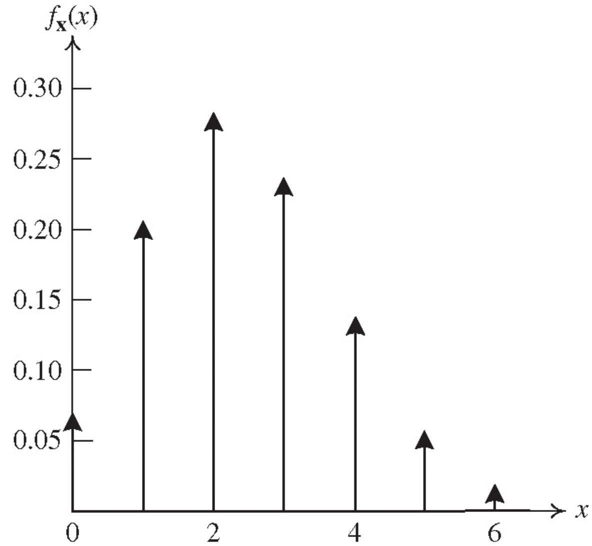
Using dirac delta functions, the binomial random variable Y has the pdf

$$f_Y(y) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(y-k)$$

Bernoulli and Binomial RVs







Gaussian Random Variables

A real-valued Gaussian random variable $X \sim N(\mu, \sigma^2)$ has the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu = \mathrm{E}[X]$ is the mean and $\sigma^2 = \mathrm{E}[(X-\mu)^2]$ is the variance. The random variable $X \sim N(0,1)$ has a **standard normal density**.
The **cumulative distribution function** (cdf) of X, $F_X(x)$, is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

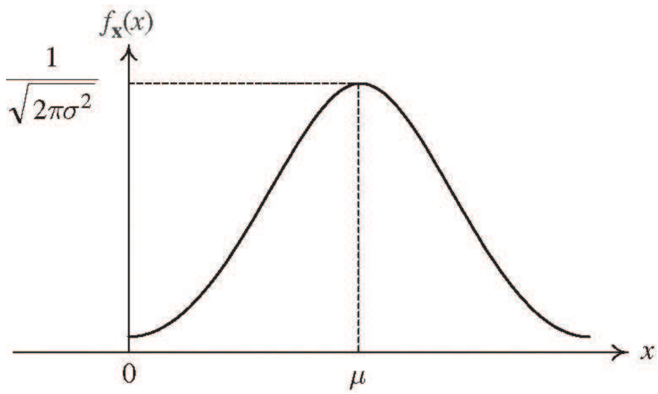
The **complementary distribution function** (cdfc), $F_X^c(x) = 1 - F_X(x)$ of a standard normal random variable defines the Gaussian Q function

$$Q(x) \triangleq \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

while its cdf defines the Gaussian Φ function

$$\Phi(x) \triangleq 1 - Q(x)$$

Gaussian RV



$f_{\mathbf{x}}(x)$
$\frac{1}{\sqrt{2\pi\sigma^2}}$
0
μ
x
$F_{\mathbf{x}}(x)$
1
$\frac{1}{2}$
0
μ
x

Gaussian Random Variables

If X is a non-standard normal random variable, $X \sim N(\mu, \sigma^2)$, then

$$\begin{aligned} F_X(x) &= \Phi\left(\frac{x-\mu}{\sigma}\right) \\ F_X^c(x) &= Q\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

The **error function** $\mathrm{erf}(x)$ and the **complementary error function** $\mathrm{erfc}(x)$, are defined by

$$\mathrm{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}dy \qquad \mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$$

Note that $\mathrm{erfc}(x) \neq 1 - \mathrm{erf}(x)$.
The complementary error function and Q function are related as follows

$$\begin{aligned} \mathrm{erfc}(x) &= 2Q(\sqrt{2}x) \\ Q(x) &= \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \end{aligned}$$

Multivariate Gaussian Distribution

Let $X_i \sim N(\mu_i, \sigma_i^2), i = 1, \ldots, n$, be correlated real-valued Gaussian random variables having covariances

$$\begin{aligned} \mu_{X_i X_j} &= \mathrm{E}\left[(X_i - \mu_i)(X_j - \mu_j)\right] \\ &= \mathrm{E}\left[X_i X_j\right] - \mu_i \mu_j, \quad 1 \leq i, j \leq n \end{aligned}$$

Let

$$\begin{aligned} \mathbf{X} &= (X_1, X_2, \ldots, X_n)^T \\ \mathbf{x} &= (x_1, x_2, \ldots, x_n)^T \\ \boldsymbol{\mu}_X &= (\mu_1, \mu_2, \ldots, \mu_n)^T \\ \boldsymbol{\Lambda} &= \begin{bmatrix} \mu_{X_1 X_1} & \cdot & \cdot & \cdot & \cdot & \mu_{X_1 X_n} \\ \vdots & & & & & \vdots \\ \mu_{X_n X_1} & \cdot & \cdot & \cdot & \cdot & \mu_{X_n X_n} \end{bmatrix} \end{aligned}$$

where $\mathbf{X}^T$ is the transpose of $\mathbf{X}$.

Multivariate Gaussian Distribution

The joint pdf of $\mathbf{X}$ defines the multivariate Gaussian distribution

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{\Lambda}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)^T \mathbf{\Lambda}^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right\}$$

where $|\mathbf{\Lambda}|$ is the determinant of $\mathbf{\Lambda}$.

Bivariate Gaussian Distribution

For the case of 2 Gaussian random variables

$$\begin{aligned} \boldsymbol{\mu}_X &= (\mu_1, \mu_2)^T \\ \boldsymbol{\Lambda} &= \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \end{aligned}$$

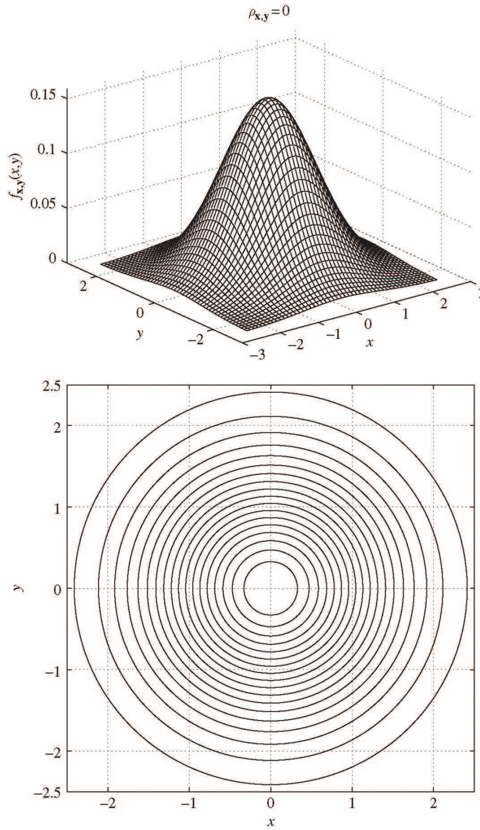
where $\rho = \mu_{12}/(\sigma_1\sigma_2) = \mu_{12}/\sigma^2$. Then $|\boldsymbol{\Lambda}| = \sigma^4(1-\rho^2)$ and

$$\boldsymbol{\Lambda}^{-1} = \frac{\sigma^2}{|\boldsymbol{\Lambda}|}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} = \frac{1}{\sigma^2(1-\rho^2)}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

With $\boldsymbol{\mu}_x = (0,0)$ we have

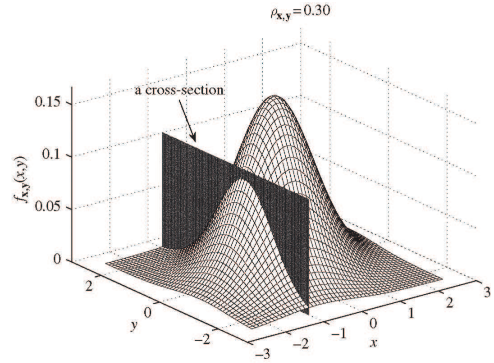
$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma^2(1-\rho^2)}(x_1,x_2)\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \\ &= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2\sigma^2(1-\rho^2)}\right] \end{aligned}$$

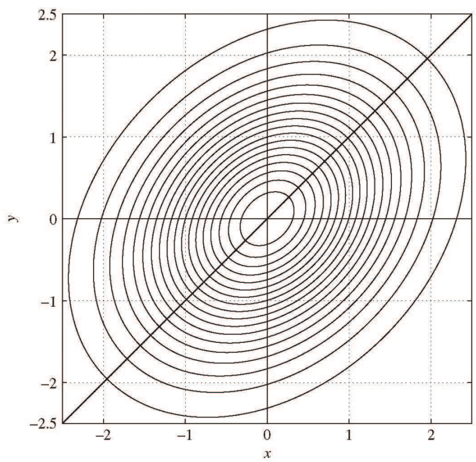
Bivariate Gaussian Distribution



$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0$.

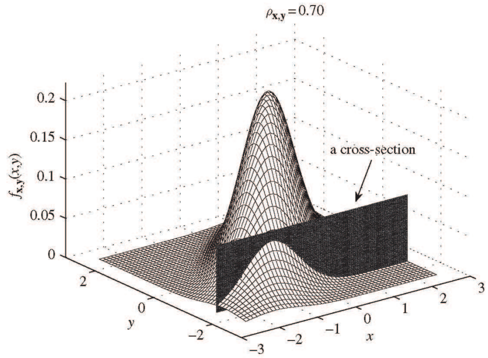
Bivariate Gaussian Distribution

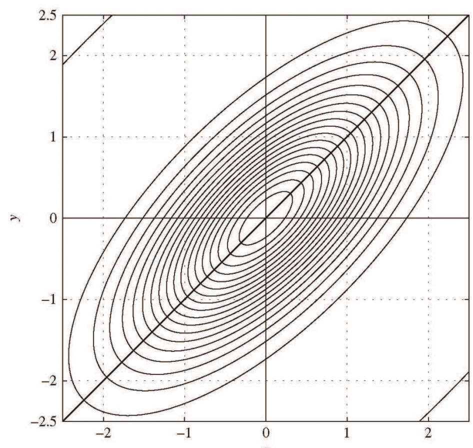




$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.3$.

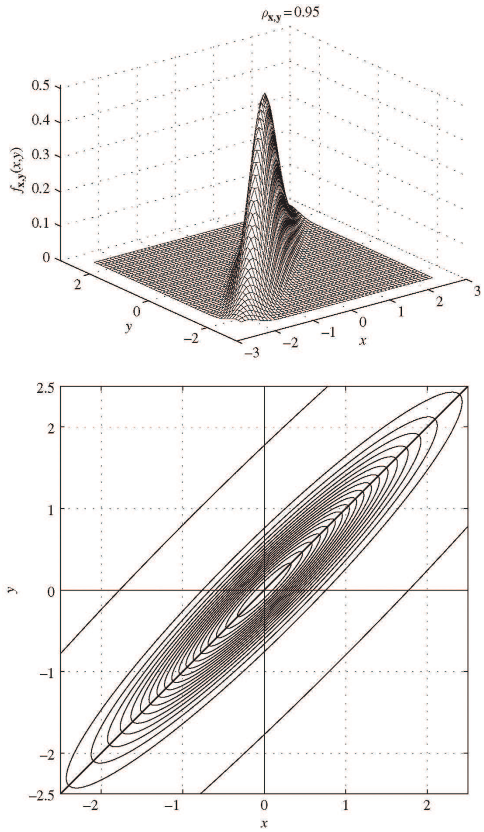
Bivariate Gaussian Distribution





$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.7$.

Bivariate Gaussian Distribution



$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.95$.

Examples

Suppose that $X \sim N(\sqrt{E}, N_o/2)$. What is the probability that $X < 0$?
Answer:

$$\begin{aligned} P(X<0) &= P(X > 2\sqrt{E}) \\ &= Q\left(\frac{2\sqrt{E}-\mu_X}{\sigma_X}\right) \\ &= Q\left(\frac{\sqrt{E}}{\sqrt{N_o/2}}\right) \\ &= Q\left(\sqrt{\frac{2E}{N_o}}\right) \end{aligned}$$

The first line follows from the fact that the pdf of X is symmetric about its mean $\sqrt{E}$.

Examples

Suppose that X and Y are independent identically distributed Gaussian random variables with mean $\sqrt{E}$ and variance $N_o/2$. What is the probability of the joint event that $X < 0$ and $Y < 0$.

Answer:

$$\begin{aligned} P(X < 0, Y < 0) &= P(X > 2\sqrt{E}, Y > 2\sqrt{E}) \\ &= P(X > 2\sqrt{E})P(Y > 2\sqrt{E}) \\ &= Q^2\left(\sqrt{\frac{2E}{N_o}}\right) \end{aligned}$$

The second line follows from the fact that X and Y are independent.

Examples

Suppose that X and Y are independent identically distributed Gaussian random variables with mean μ and variance σ^2. What is the mean and variance of the random variable XY.
Answer: We could use the joint pdf $f_{XY}(x,y)$ and integrate, viz.,

$$\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy$$

However, there is a much easier approach

$$\mu_{XY} = \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y] = \mu_X \mu_Y = \mu^2$$

$$\begin{aligned} \sigma^2_{XY} &= \mathrm{E}[(XY - \mu_{XY})^2] \\ &= \mathrm{E}[(XY)^2 - 2\mathrm{E}[XY]\mu_{XY} + \mu^2_{XY} \\ &= \mathrm{E}[X^2]\mathrm{E}[Y^2] - \mu^4 \\ &= (\sigma^2 + \mu^2)^2 - \mu^4 \\ &= \sigma^4 + 2\mu^2\sigma^2 \end{aligned}$$