EE4601
Communication Systems

Week 3

Random Processes, Stationarity, Means, Correlations

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Random Processes

A random process or stochastic process, $X(t)$, is an ensemble of ζ sample functions $\{X_1(t), X_2(t), \ldots, X_\zeta(t)\}$ together with a probability rule which assigns a probability to any meaningful event associated with the observation of these sample functions.

Suppose the sample function $X_i(t)$ corresponds to the sample point s_i in the sample space S and occurs with probability P_i.

- ζ may be finite or infinite.
- Sample functions may be defined at discrete or continuous time instants.
  - this defines discrete- or continuous-*time* random processes.
- Sample function values may take on discrete or continuous values.
  - this defines discrete- or continuous-*parameter* random processes.

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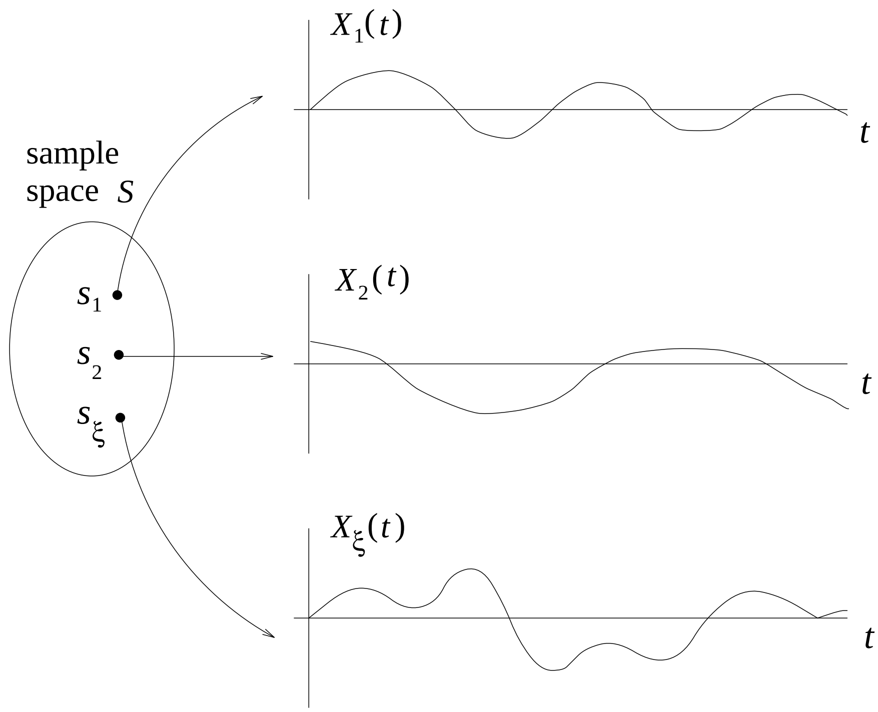
Random Processes



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Random Processes vs. Random Variables

What is the difference between random variable and processes?

- For a random variable, the outcome of a random experiment is mapped onto a *variable*, e.g., a number.
- For a random processes, the outcome of a random experiment is mapped onto a *waveform* that is a function of time.

Suppose that we observe a random process $X(t)$ at some time t_1 to generate the observation $X(t_1)$ and that the number of possible sample functions or waveforms, ζ, is finite.

If $X_i(t_1)$ is observed with probability P_i, then the collection of numbers $\{X_i(t_1)\}, i = 1, 2, \ldots, \zeta$ forms a random variable, denoted by $X(t_1)$, having the probability distribution $\mathrm{P}_i, i = 1, 2, \ldots, \zeta$.

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Random Processes

The collection of n random variables, $X(t_1), \ldots,\ X(t_n)$, has the joint cdf

$$F_{X(t_1),\ldots,\ X(t_n)}(x_1, \ldots,\ x_n) = \mathrm{P_r}(X(t_1) < x_1, \ldots, X(t_n) < x_n)\ .$$

A more compact notation can be obtained by defining the vectors

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, \ldots, x_n)^T \\ \mathbf{X}(t) &= (X(t_1), X(t_2), \ldots, X(t_n))^T \end{aligned}$$

Then the joint cdf and joint pdf of $\mathbf{X}(t)$ are, respectively,

$$\begin{aligned} F_{\mathbf{X}(t)}(\mathbf{x}) &= P(\mathbf{X}(t) \le \mathbf{x}) \\ p_{\mathbf{X}(t)}(\mathbf{x}) &= \frac{\partial^n F_{\mathbf{X}(t)}(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n} \end{aligned}$$

A random process is **strictly stationary** if and only if the equality

$$p_{\mathbf{X}(t)}(\mathbf{x}) = p_{\mathbf{X}(t+\tau)}(\mathbf{x})$$

holds for all sets of time instants $\{t_1, t_2, \ldots, t_n\}$ and all time shifts τ.

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Ensemble and Time Averages

For a random process, we define the following two operators

$$\begin{aligned} \mathrm{E}[\ \cdot\ ] &\triangleq \textbf{ensemble average} \\ <\ \cdot\ > &\triangleq \textbf{time average} \end{aligned}$$

The *ensemble mean* or *ensemble average* of a random process $X(t)$ at time t is

$$\mu_X(t) \equiv \mathrm{E}[X(t)] = \int_{-\infty}^{\infty} x p_{X(t)}(x) dx$$

The *time average mean* or *time average* of a random process $X(t)$ is

$$< X(t) >= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) dt$$

- In general, the time average mean $< X(t) >$ is also a random variable, because it depends on the particular sample function that is observed for time averaging.

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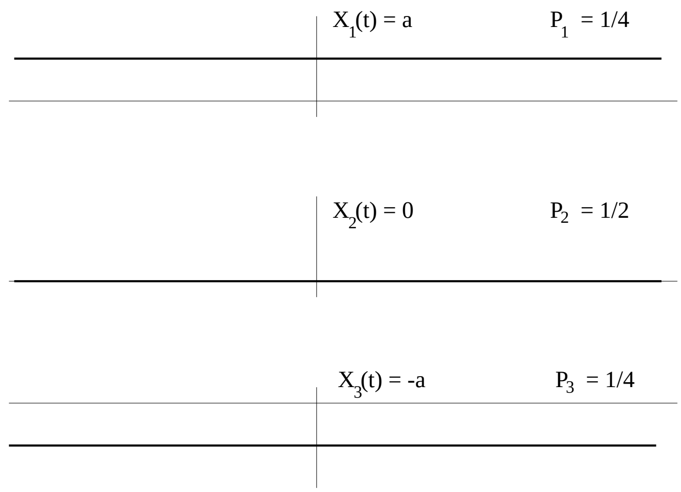
Example

Consider the random process shown below.



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Example

The *ensemble mean* is

$$\begin{aligned} \mathrm{E}[X(t)] &= X_1(t)P_1 + X_2(t)P_2 + X_3(t)P_3 \\ &= a \cdot 1/4 + 0 \cdot 1/2 + (-a) \cdot 1/4 = 0 \end{aligned}$$

The *time average mean* is

$$\langle X(t) \rangle = \begin{cases} a & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -a & \text{with probability } 1/4 \end{cases}$$

Note that $\langle X(t) \rangle$ is a random variable (since it depends on the sample function that is chosen for time averaging, while $\mathrm{E}[X(t)]$ is just a number (that in the above example is not a function of time t, but in general may a function of the time variable t).

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Moments and Correlations

E[$\cdot$] = ensemble average operator.

[Ensemble] Mean: $\mu_X(t_1) = \mathrm{E}[X(t_1)] = \int_{-\infty}^{\infty} x f_{X(t_1)}(x)dx$

[Ensemble] Variance: $\sigma_X^2(t_1) = \mathrm{E}[(X(t_1) - \mu_X(t_1))^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_{X(t_1)}(x)dx$

[Ensemble] Autocorrelation: $\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$

[Ensemble] Autocovariance:

$$\begin{aligned} \mu_{XX}(t_1, t_2) &= \mathrm{E}[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] \\ &= \phi_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned}$$

If $X(t)$ has zero mean, then $\mu_{XX}(t_1, t_2) = \phi_{XX}(t_1, t_2)$.

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Example

Consider the random process

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_c are constants. The phase Θ is assumed to be a uniformly distributed random variable with pdf

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi)\ , & 0 \le \theta \le 2\pi \\ 0\ , & \text{elsewhere} \end{cases}$$

The ensemble mean of $X(t_1)$ is obtained by averaging over the pdf of Θ:

$$\begin{aligned} \mu_X(t_1) &= \mathrm{E}_\Theta[X(t_1)] = \mathrm{E}_\Theta[A\cos(2\pi f_c t_1 + \Theta)] \\ &= \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi f_c t_1 + \theta) d\theta \\ &= \frac{A}{2\pi}\sin(2\pi f_c t_1 + \theta)\Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

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Example (cont'd)

The autocorrelation of $X(t) = A\cos(2\pi f_c t + \Theta)$ is

$$\begin{aligned}
\phi_{XX}(t_1, t_2) &= \mathrm{E}_\Theta[X(t_1)X(t_2)] \\
&= \mathrm{E}_\Theta[A^2 \cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)] \\
&= \frac{A^2}{2}\mathrm{E}_\Theta[\cos(2\pi f_c t_1 + 2\pi f_c t_2 + 2\Theta)] + \frac{A^2}{2}\mathrm{E}_\Theta[\cos(2\pi f_c(t_1 - t_2)]
\end{aligned}$$

But

$$\begin{aligned}
\mathrm{E}_\Theta[\cos(2\pi f_c t_1 + 2\pi f_c t_2 + 2\Theta)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi f_c t_1 + 2\pi f_c t_2 + 2\theta) d\theta \\
&= \frac{1}{4\pi}\sin(2\pi f_c t_1 + 2\pi f_c t_2 + 2\theta) d\theta\Big|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

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Example (cont'd)

Also,

$$\mathrm{E}_\Theta[\cos(2\pi f_c(t_1 - t_2)] = \cos 2\pi f_c(t_1 - t_2)$$

Hence,

$$\begin{aligned}\phi_{XX}(t_1, t_2) &= \frac{A^2}{2}\cos 2\pi f_c(t_1 - t_2) \\ &= \frac{A^2}{2}\cos 2\pi f_c\tau, \quad \tau = t_1 - t_2\end{aligned}$$

The autocovariance of $X(t)$ is

$$\begin{aligned}\mu_{XX}(t_1, t_2) &= \phi_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\ &= \phi_{XX}(\tau)\end{aligned}$$

since $\mu_X(t) = 0$.

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Wide Sense Stationary

A **wide sense stationary** random process $X(t)$ has the property

$$\begin{aligned} \mu_X(t) &= \mu_X \quad \text{a constant} \\ \phi_X(t_1, t_2) &= \phi_X(\tau) \quad \text{where } \tau = t_2 - t_1 \end{aligned}$$

The autocorrelation function only depends on the time difference τ.

If a random process is strictly stationary, then it is wide sense stationary.
The converse is not true.

$$\text{strictly stationary} \longrightarrow \text{wide sense stationary}$$

For a Gaussian random process only

$$\text{strictly stationary} \longleftrightarrow \text{wide sense stationary}$$

The previous example is a wide sense stationary random process.

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Some Properties of $\phi_{XX}(\tau)$

The autocorrelation function, $\phi_{XX}(\tau)$, of a wide sense stationary random process $X(t)$ satisfies the following properties.

1. $\phi_{XX}(0) = \mathrm{E}[X^2(t)]$: total power ac + dc
2. $\phi_{XX}(\tau) = \phi_{XX}(-\tau)$: even function
3. $|\phi_{XX}(\tau)| \leq \phi_{XX}(0)$: a variant of the Cauchy-Schwartz inequality. Proof on next slide.
4. $\phi_{XX}(\infty) = \mathrm{E}^2[X(t)] = \mu_X^2$: dc power, if $X(t)$ has no periodic components.
5. If $p_{X(t)}(x) = p_{X(t+T)}(x)$, i.e., the pdf of $X(t)$ is periodic in t with period T, then $\phi_{XX}(\tau) = \phi_{XX}(\tau + T)$. In other words, if $p_{X(t)}(x)$ is periodic in t with period T, then $\phi_{XX}(\tau)$ is periodic in τ with period T. Such a random process is said to be **periodic wide sense stationary** or **cyclostationary**. Digitally modulated waveforms are cyclostationary random processes.

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Some Properties of $\phi_{XX}(\tau)$

The inequality $|\phi_{XX}(\tau)| \leq \phi_{XX}(0)$ can be established through the following steps.

$$
\begin{aligned}
0 &\leq \mathrm{E}[(X(t+\tau) \pm X(t))^2] \\
&= \mathrm{E}[X^2(t) + X^2(t+\tau) \pm X(t+\tau)X(t)] \\
&= \mathrm{E}[X^2(t)] + \mathrm{E}[X^2(t+\tau)] \pm \mathrm{E}[X(t+\tau)X(t)] \\
&= 2\mathrm{E}[X^2(t)] \pm \mathrm{E}[X(t+\tau)X(t)] \\
&= 2\phi_{XX}(0) \pm 2\phi_{XX}(\tau)\ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pm\phi_{XX}(\tau) &\leq \phi_{XX}(0) \\
|\phi_{XX}(\tau)| &\leq \phi_{XX}(0)\ .
\end{aligned}
$$

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