EE4601
Communication Systems

Week 4

Ergodic Random Processes, Power Spectrum
Linear Systems

Ergodic Random Processes

An **ergodic** random process is one where time averages are equal to ensemble averages. Hence, for all $g(\mathbf{X})$ and $\mathbf{X}$

$$\begin{aligned}
\mathrm{E}[g(\mathbf{X})] &= \int_{-\infty}^{\infty} g(\mathbf{X}) p_{\mathbf{X}(t)}(\mathbf{x}) d\mathbf{x} \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g[\mathbf{X}(t)] \mathrm{d}t \\
&= < g[\mathbf{X}(t)] >
\end{aligned}$$

For a random process to be ergodic, it must be strictly stationary. However, not all strictly stationary random processes are ergodic.

A random process is **ergodic in the mean** if

$$< X(t) > = \ \mu_X$$

and **ergodic in the autocorrelation** if

$$< X(t)X(t+\tau) > = \ \phi_{XX}(\tau)$$

Example (cont'd)

Recall the random process

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_c are constants, and Θ is assumed to be a uniformly distributed random phase having the pdf

$$p_\Theta(\theta) = \begin{cases} 1/(2\pi)\ , & 0 \le \theta \le 2\pi \\ 0\ , & \text{elsewhere} \end{cases}$$

The time average mean of $X(t)$ is

$$< X(t) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_c t + \theta)dt = 0$$

In this example $\mu_X(t) = \mathrm{E}[X(t)] =< X(t) >= 0$, so the random process $X(t)$ is ergodic in the mean.

N.B. Make sure you understand the difference between the *time average* and *ensemble average*.

Example (cont'd)

The time average autocorrelation of $X(t)$ is

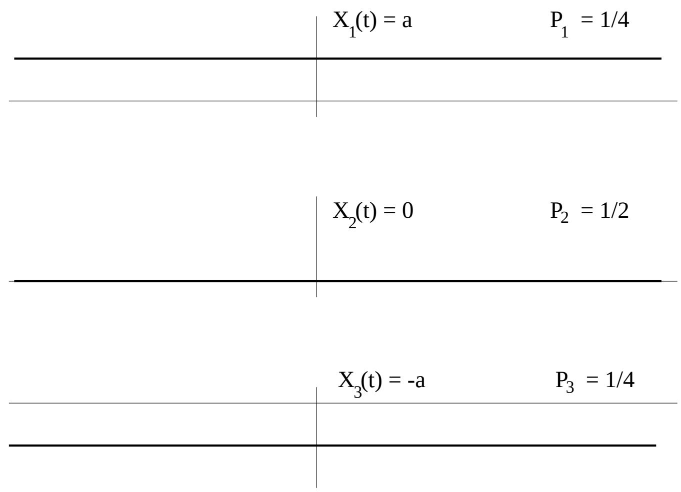
$$\begin{aligned} < X(t)X(t+\tau) > &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta) dt \\ &= \lim_{T\to\infty} \frac{A^2}{4T}\int_{-T}^{T} \left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right] dt \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau) \end{aligned}$$

In this example $\phi_X(\tau) = \mathrm{E}[X(t)X(t+\tau)] \; = < X(t)X(t+\tau) >$, so the random process $X(t)$ is ergodic in the autocorrelation.

It follows that the random process $X(t)$ in this example is *ergodic in the mean and autocorrelation.*

Example

Consider the random process shown below.



Example (cont'd)

For this example, the *ensemble* and *time average* means are, respectively,

$$\begin{aligned}\mu_X &= \mathrm{E}[X(t)] = 0 \\ \langle X(t) \rangle &= \begin{cases} a & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -a & \text{with probability } 1/4 \end{cases}\end{aligned}$$

Hence, $X(t)$ is *not ergodic in the mean.*

The *ensemble* and *time average* autocorrelations are

$$\begin{aligned}\phi_{XX}(\tau) &= \mathrm{E}[X(t)X(t+\tau)] = a^2(1/4) + 0(1/2) + (-a)^2(1/4) = a^2/2 \\ \langle X(t)X(t+\tau) \rangle &= \begin{cases} a^2 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases}\end{aligned}$$

Hence, $X(t)$ is *not ergodic in the autocorrelation.*

Example (cont'd)

Note that

$$\begin{aligned} \mathrm{E}[\langle X(t) \rangle] &= \mu_X \\ \mathrm{E}[\langle X(t)X(t+\tau) \rangle] &= \phi_{XX}(\tau) \end{aligned}$$

Because of this property $\langle X(t) \rangle$ and $\langle X(t)X(t+\tau) \rangle$ are said to provide *unbiased estimates* of μ_X and $\phi_{XX}(\tau)$, respectively.

Power Spectral Density

The power spectral density (psd) of a wide sense stationary random process $X(t)$ is the Fourier transform of its autocorrelation function, i.e.,

$$\begin{aligned} \Phi_{XX}(f) &= = \int_{-\infty}^{\infty} \phi_{XX}(\tau) e^{-j2\pi f\tau} d\tau \\ \phi_{XX}(\tau) &= \int_{-\infty}^{\infty} \Phi_{XX}(f) e^{j2\pi f\tau} df \ . \end{aligned}$$

We have seen that $\phi_{XX}(\tau)$ is real and even. Therefore, $\Phi_{XX}(-f) = \Phi_{XX}(f)$ meaning that $\Phi_{XX}(f)$ is also real and even.

The total power (ac + dc), P, in a random process $X(t)$ is

$$P = \mathrm{E}[X^2(t)] = \phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) df$$

a famous result known as **Parseval's theorem**.

Example

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_v are constants and

$$p_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}\ , & -\pi \le \theta \le \pi \\ 0\ , & \text{elsewhere} \end{cases}$$

We have seen before that

$$\phi_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Hence,

$$\begin{aligned} \Phi_{XX}(f) &= \frac{A^2}{2}\mathcal{F}[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{4}\left(\delta(f - f_c) + \delta(f + f_c)\right) \end{aligned}$$

Properties of $\Phi_{XX}(f)$

1. $\Phi_{XX}(0) = \int_{-\infty}^{\infty} \phi_{XX}(\tau)d\tau$
2. $\int_{0-}^{0+} \Phi_{XX}(f)df =$ dc power
3. $\phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f)df =$ total power
4. $\Phi_{XX}(f) \geq 0$ for all f. Power is never negative.
5. $\Phi_{XX}(f) = \Phi_{XX}(-f)$ (even function) if $X(t)$ is a real random process.
6. $\Phi_{XX}(f)$ is always real.

Discrete-time Random Processes

Consider a discrete-time real-valued random process X_n, that consists of an ensemble of discrete-time sample sequences $\{x_n\}$.
The ensemble mean of X_n is

$$\mu_{X_n} = \mathrm{E}[X_n] = \int_{-\infty}^{\infty} x_n f_{X_n}(x_n) dx_n$$

The ensemble autocorrelation of X_n is

$$\phi_{XX}(n,k) = \mathrm{E}[X_n X_k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_n X_k f_{X_n,X_k}(x_n,x_k) dx_n dx_k$$

For a wide-sense stationary discrete-time real-valued random process, we have

$$\begin{aligned} \mu_{X_n} &= \mu_X, \quad \forall n \\ \phi_{XX}(n,k) &= \phi_{XX}(n-k) \end{aligned}$$

From Parseval's theorem, the total power in the process X_n is

$$P = \mathrm{E}[X_n^2] = \phi_{XX}(0)$$

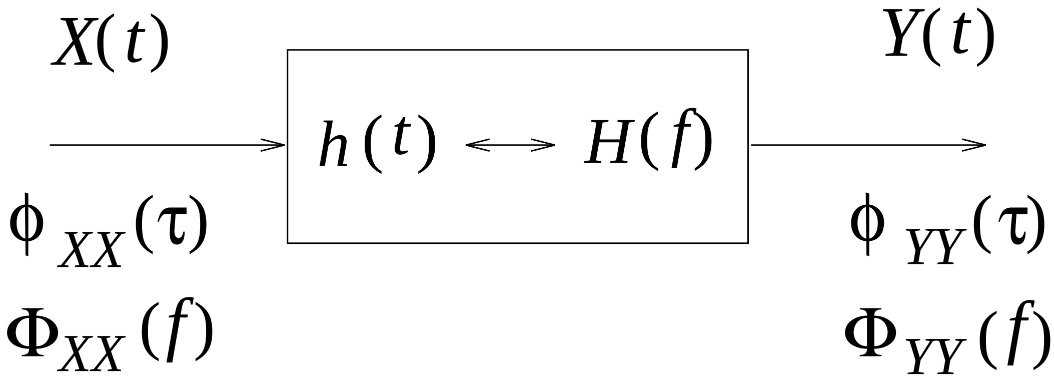
Power Spectrum of Discrete-time RP

The power spectrum of the real-valued wide-sense stationary discrete-time random process X_n is the discrete-time Fourier transform of its autocorrelation function, i.e.,

$$\begin{aligned}\Phi_{XX}(f) &= \sum_{n=-\infty}^{\infty} \phi_{XX}(n) e^{-j2\pi fn} \\ \phi_{XX}(n) &= \int_{-1/2}^{1/2} \Phi_{XX}(f) e^{j2\pi fn} df\end{aligned}$$

Observe that the power spectrum $\Phi_{XX}(f)$ is periodic in frequency f with a period of unity. In other words $\Phi_{XX}(f) = \Phi_{XX}(f+k)$, for $k = \pm 1, \pm 2, \ldots$ This is a characteristic of any discrete-time sequence. For example, one obtained by sampling a continuous-time random process $X_n = x(nT_s)$, where T_s is the sample period.

Linear Systems



Linear Systems

Suppose that the input to the linear system (filter) $h(t)$ is a wide sense stationary random process $X(t)$, with mean μ_X and autocorrelation $\phi_{XX}(\tau)$.

The input and output waveforms are related by the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \ \ .$$

Hence,

$$Y(f) = H(f)X(f) \ \ .$$

The output mean is

$$\mu_Y = \int_{-\infty}^{\infty} h(\tau)\mathrm{E}[X(t-\tau)]d\tau = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau = \mu_X H(0) \ \ .$$

The mean value of the filter output (dc output) is just the mean value of the filter input (dc input) multiplied by the dc gain of the filter.

Linear Systems

The output autocorrelation is

$$\begin{aligned}
\phi_{YY}(\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\beta)X(t-\beta)d\beta \int_{-\infty}^{\infty} h(\alpha)X(t+\tau-\alpha)d\alpha\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\mathrm{E}\left[X(t-\beta)X(t+\tau-\alpha)\right]d\beta d\alpha \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\phi_{XX}(\tau-\alpha+\beta)d\beta d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta-\alpha)d\alpha d\beta \\
&= \left\{\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta)d\beta\right\} * h(\tau) \\
&= h(-\tau) * \phi_{XX}(\tau) * h(\tau)\ .
\end{aligned}$$

Taking transforms, the output psd is

$$\begin{aligned}
\Phi_{YY}(f) &= H^*(f)\Phi_{XX}(f)H(f) \\
&= |H(f)|^2\,\Phi_{XX}(f)\ .
\end{aligned}$$

Cross-correlation and Cross-covariance

If $X(t)$ and $Y(t)$ are each wide sense stationary and jointly wide sense stationary, then

$$\begin{aligned}\boldsymbol{\phi}_{XY}(t, t+\tau) &= \mathrm{E}[X(t)Y(t+\tau)] = \boldsymbol{\phi}_{XY}(\tau) \\ \boldsymbol{\mu}_{XY}(t, t+\tau) &= \boldsymbol{\mu}_{XY}(\tau) = \boldsymbol{\phi}_{XY}(\tau) - \mu_x \mu_y\end{aligned}$$

The crosscorrelation function $\phi_{XY}(\tau)$ has the following properties.

1. $\phi_{XY}(\tau) = \phi_{YX}(-\tau)$
2. $|\phi_{XY}(\tau)| \leq \frac{1}{2}[\phi_{XX}(0) + \phi_{YY}(0)]$
3. $|\phi_{XY}(\tau)|^2 \leq \phi_{XX}(0)\phi_{YY}(0)$ if $X(t)$ and $Y(t)$ have zero mean.

Example

Consider the linear system shown in the previous example. The crosscorrelation between the input process $X(t)$ and the output process $Y(t)$ is

$$\begin{aligned}\phi_{YX}(\tau) &= \mathrm{E}[Y(t)X(t+\tau)] \\ &= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha X(t+\tau)\right] \\ &= \int_{-\infty}^{\infty} h(\alpha)\mathrm{E}\left[X(t-\alpha)X(t+\tau)\right]d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha)\phi_{XX}(\tau+\alpha)d\alpha \\ &= h(-\tau)*\phi_{XX}(\tau)\end{aligned}$$

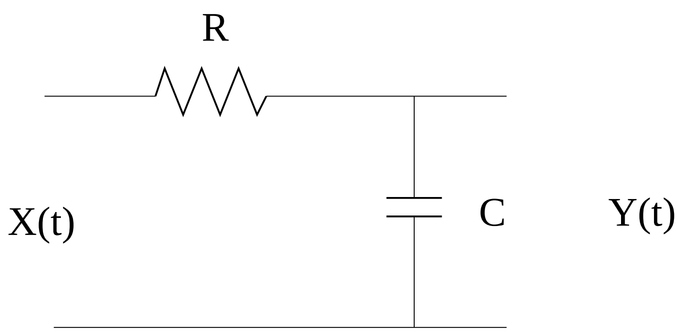
The cross power spectral density is

$$\Phi_{YX}(f) = H^*(f)\Phi_{XX}(f)$$

Note also that

$$\phi_{YX}(-\tau) = \phi_{XY}(\tau)$$

Example



Example

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Suppose $X(t)$ has autocorrelation function $\phi_{XX}(\tau) = e^{-\alpha|\tau|}$. What is $\phi_{YY}(\tau)$?

We have

$$\Phi_{YY}(f) = |H(f)|^2 \Phi_{XX}(f)$$

where

$$\begin{aligned} |H(f)|^2 &= \frac{1}{1 + (2\pi fRC)^2} \\ \Phi_{XX}(f) &= \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \end{aligned}$$

Hence,

$$\Phi_{YY}(f) = \frac{1}{1 + (2\pi fRC)^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Example

Do you remember *partial fractions*? Now you need them!
We write

$$\Phi_{YY}(f) = \frac{A}{\alpha^2 + (2\pi f)^2} + \frac{B}{1 + (2\pi f RC)^2}$$

and solve for A and B. We have

$$A(1 + (2\pi f RC)^2) + B(\alpha^2 + (2\pi f)^2) = 2\alpha$$

Clearly,

$$\begin{aligned} A + B\alpha^2 &= 2\alpha \\ A(2\pi f RC)^2 + B(2\pi f)^2 &= 0 \end{aligned}$$

From the second equation

$$A = -\frac{B}{(RC)^2} = -B\beta^2$$

where $\beta = 1/(RC)$.

Example

Then using the first equation

$$B = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Also,

$$A = -B\beta^2 = -\frac{2\alpha\beta^2}{\alpha^2 - \beta^2}$$

Finally,

$$\Phi_{YY}(f) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot \frac{2\beta}{\beta^2 + (2\pi f)^2}$$

Now take inverse Fourier transforms to get

$$\phi_{YY}(\tau) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot e^{-\alpha|\tau|} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot e^{-\beta|\tau|}$$

Discrete-time Random Processes

Consider a wide-sense stationary discrete-time random process X_n that is input to a discrete-time linear time-invariant filter having impulse response h_n. The frequency response function of the filter is the discrete time Fourier transform

$$H(f) = \sum_{n=-\infty}^{\infty} h_n e^{-j2\pi fn}$$

The output of the filter is the convolution sum

$$Y_k = \sum_{n=-\infty}^{\infty} h_n X_{k-n}$$

It follows that the output mean is

$$\begin{aligned} \mu_Y = \mathrm{E}[Y_k] &= \sum_{n=-\infty}^{\infty} h_n \mathrm{E}[X_{k-n}] \\ &= \mu_X \sum_{n=-\infty}^{\infty} h_n \\ &= \mu_X H(0) \end{aligned}$$

Discrete-time Random Processes

The autocorrelation function of the output process is

$$\begin{aligned}
\phi_{YY}(k) &= \mathrm{E}[Y_n Y_{n+k}] \\
&= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \mathrm{E}[X_{n-i} h_j X_{n+k-j}] \\
&= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \phi_{XX}(k-j+i)]
\end{aligned}$$

By taking the discrete-time Fourier transform of $\phi_{YY}(k)$ and using the above relationship, we can obtain

$$\Phi_{YY}(f) = \Phi_{XX}(f)|H(f)|^2$$

Again, note in this case that $\Phi_{YY}(f)$ is periodic in f with a period of unity.