EE4601
Communication Systems

Week 5
Noise and Matched Filters

Error Probability with Binary Signaling

Thermal Noise

Thermal noise affect all communication receivers.

From fundamental physics (which we will not go into here) the power spectral density of thermal noise is

$$\Phi_{nn}(f) = \frac{h|f|}{2\left(e^{h|f|/kT} - 1\right)} \text{ watts/Hz}$$

where

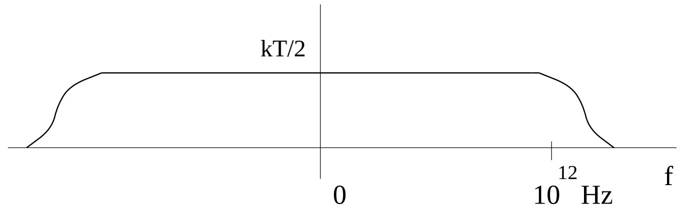
$$\begin{aligned} h &= 6.62 \times 10^{-34} \text{ Joules} = \text{Plank's constant} \\ k &= 1.37 \times 10^{-23} \text{ Joules/degree} = \text{Boltzmann's constant} \end{aligned}$$

Using the Taylor series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ gives

$$\begin{aligned} \Phi_{nn}(f) &\approx \frac{h|f|}{2\left(1 + h|f|/kT - 1\right)} \\ &= \frac{kT}{2} \text{ watts/Hz} \end{aligned}$$

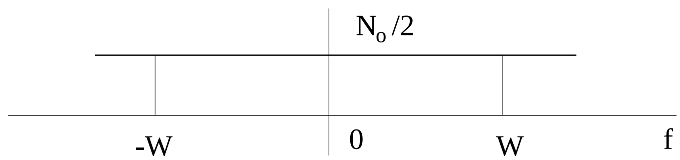
White Noise

$\Phi_{nn}(f)$



Over a narrow bandwidth of frequencies the noise spectral density can be considered "flat"

$\Phi_{nn}(f)$



White Noise

If we assume the bandwidth W is infinite (idealization), then the autocorrelation function of the zero-mean additive white Gaussian noise is

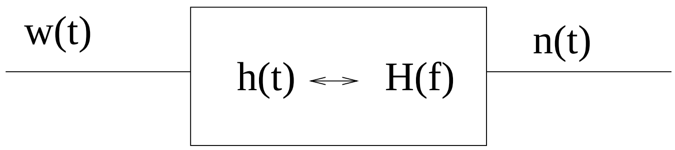
$$\phi_{ww}(\tau) = \mathcal{F}^{-1}\left\{N_o/2\right\} = \frac{N_o}{2}\delta(\tau)$$

where we use a subscript "w" to emphasize that the noise is white.
Observe that $w(t)$ is *uncorrelated* with $w(t+\tau)$ and since Gaussian, statistically *independent* for any $\tau \neq 0$.

The noise power in bandwidth W is

$$P_n = 2 \times W \times \frac{N_o}{2} = N_o W \quad \text{watts}$$

Filtered White Noise



If the input noise spectral density is $\Phi_{ww}(f) = N_o/2$, then the output noise spectral density is

$$\Phi_{nn}(f) = \frac{N_o}{2}\left|H(f)\right|^2$$

For example, consider the ideal low-pass filter

$$H(f) = \mathrm{rect}\left(\frac{f}{2W}\right) = \begin{cases} 1, & |f| \leq W \\ 0, & \text{elsewhere} \end{cases}$$

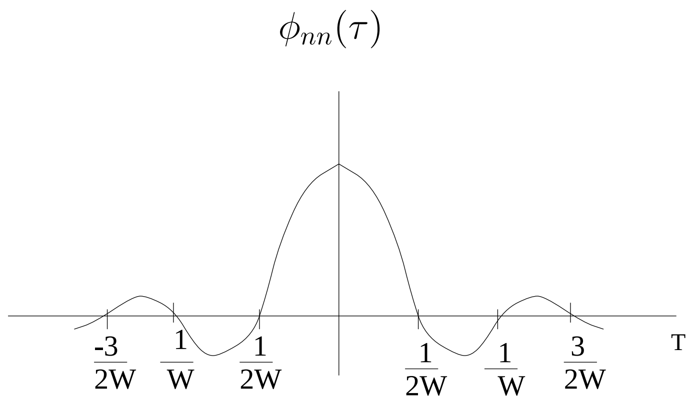
Then

$$\Phi_{nn}(f) = \frac{N_o}{2}\mathrm{rect}\left(\frac{f}{2W}\right)$$

Filtered White Noise

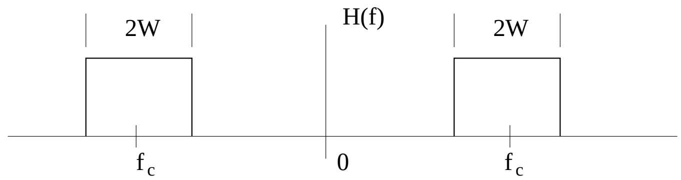
The autocorrelation function of the ideal low-pass filtered noise is

$$\begin{aligned}\phi_{nn}(\tau) &= \frac{N_o}{2} 2W \mathrm{sinc}(2W\tau) \\ &= N_o W \mathrm{sinc}(2W\tau)\end{aligned}$$



Observe that samples of $n(t)$ taken $1/(2W)$ seconds apart, or any multiple of $1/(2W)$ seconds are *uncorrelated*. If the noise is Gaussian, then the samples are statistically independent.

Bandpass Filtered White Noise



$$H(f) = \mathrm{rect}\left(\frac{f - f_c}{2W}\right) + \mathrm{rect}\left(\frac{f + f_c}{2W}\right)$$

$$\begin{aligned}
\Phi_{nn}(f) &= \frac{N_o}{2}\left[\mathrm{rect}\left(\frac{f - f_c}{2W}\right) + \mathrm{rect}\left(\frac{f + f_c}{2W}\right)\right] \\
\phi_{nn}(\tau) &= \frac{N_o}{2} \cdot 2W \mathrm{sinc}(2W\tau) \cdot 2\cos 2\pi f_c \tau \\
&= 2N_o W \mathrm{sinc}(2W\tau) \cdot \cos 2\pi f_c \tau
\end{aligned}$$

Noise Equivalent Bandwidth

Consider an arbitatry filter with tranfer function $H(f)$. If the input to the filter is white noise with power spectral density $N_o/2$, then the noise power at the output of the filter is

$$\begin{aligned} N_{\text{out}} &= \frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2 df \\ &= N_o\int_{0}^{\infty}|H(f)|^2 df \end{aligned}$$

Next suppose that the same noise process is applied to an ideal low-pass filter with bandwidth B and zero frequency response $H(0)$. The noise at the output of the filter is

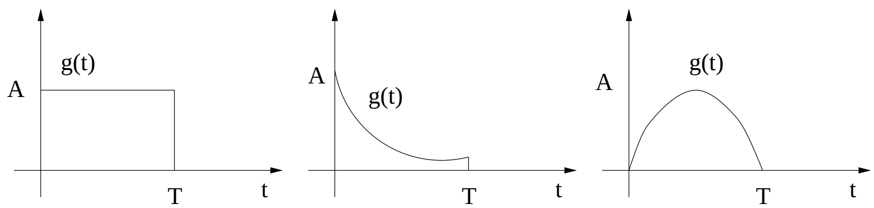
$$N_{\text{out}} = N_o B H^2(0)$$

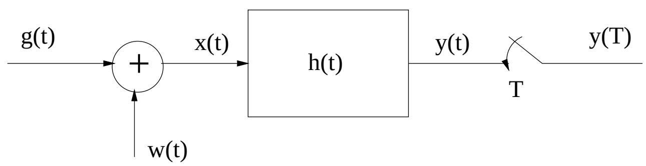
Equating the above two equations give the *noise equivalent bandwdith*

$$B = \frac{\int_0^\infty |H(f)|^2 df}{H^2(0)}$$

Basic Problem

A pulse $g(t)$ is transmitted over a noisy channel, representing a "0" or "1". The pulse is assumed to have duration T.





$\Phi_{ww}(f) = \frac{N_o}{2}$
Given the knowledge of $g(t)$, how do we choose $h(t)$ to minimize the effects of noise?

Matched Filter

$$y(t) = g_o(t) + n(t)$$

where

$$\begin{aligned} g_o(t) &= g(t) * h(t) \\ n(t) &= w(t) * h(t) \end{aligned}$$

We wish to maximize the <u>peak pulse signal-to-noise ratio</u>

$$\eta = \frac{|g_o(T)|^2}{E[n^2(T)]} = \frac{\text{instantaneous signal power}}{\text{average noise power}}$$

where $T =$ sampling instant.
We have $\Phi_{nn}(f) = |H(f)|^2 \Phi_{ww}(f) = \frac{N_o}{2}|H(f)|^2$

$$\begin{aligned} E[n^2(T)] &= \phi_{nn}(0) = \int_{-\infty}^{\infty} \Phi_{nn}(f) df = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \\ g_o(T) &= \int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi f T} df \end{aligned}$$

Matched Filter

Then,

$$\eta = \frac{|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df|^2}{\frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2 df}$$

Choose $H(f)$ to maximize η.
Apply the <u>Schwartz</u> inequality

$$|\int_{-\infty}^{\infty} x(f)y(f)df|^2 \leq \int_{-\infty}^{\infty}|x(f)|^2 df \int_{-\infty}^{\infty}|y(f)|^2 df$$

with equality iff $x(f) = ky^*(f)$, k - arbitrary scalar constant
Hence,

$$\left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2 \leq \int_{-\infty}^{\infty}|G(f)|^2 df \int_{-\infty}^{\infty}|H(f)|^2 df$$

and

$$\eta \leq \frac{2}{N_o}\int_{-\infty}^{\infty}|G(f)|^2 df$$

Since the RHS does not depend on $H(f)$, we maximize η by choosing

$$H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi fT} \leftrightarrow kg(T-t) = h_{\text{opt}}(t)$$

Matched Filter

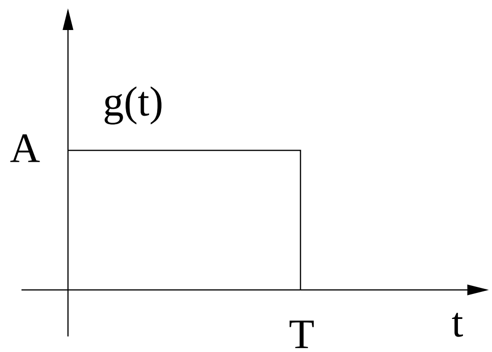
This gives

$$\eta_{\max} = \frac{2}{N_o}\int_{-\infty}^{\infty} |G(f)|^2 df = \frac{E}{N_o/2}$$

Recall Rayleigh's energy theorem

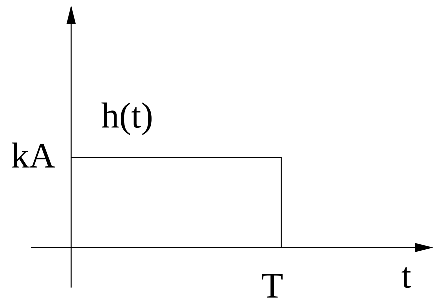
$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Example: $g(t) = Au_T(t) = A\left(u(t) - u(t-T)\right)$

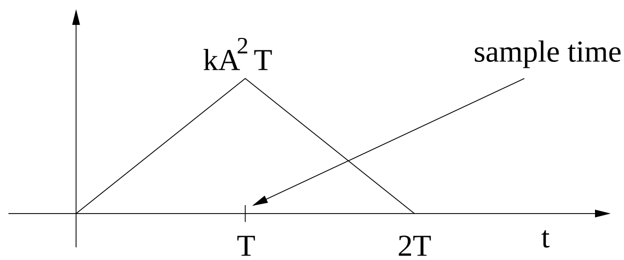


Matched Filter

$$h(t) = kg(T - t)$$



$$g_o(t) = g(t) * h(t)$$



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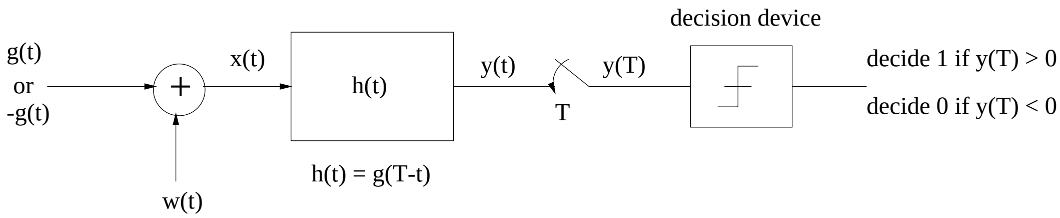
Binary Signaling

Antipodal signaling

$$
\begin{aligned}
`1' &\rightarrow g(t) \\
`0' &\rightarrow -g(t)
\end{aligned}
$$



Assume $g(t)$ was sent, i.e., '1' was sent

$$y(t) = \int_0^t x(\alpha)h(t-\alpha)d\alpha$$

Note

$$
\begin{aligned}
h(t-\alpha) &= g(T-t+\alpha) \\
h(T-\alpha) &= g(\alpha)
\end{aligned}
$$

Binary Signaling

$$\begin{aligned} y(T) &= \int_0^T x(\alpha)h(T-\alpha)d\alpha \\ &= \int_0^T [g(\alpha)+w(\alpha)]g(\alpha)d\alpha \\ &= \int_0^T g^2(\alpha)d\alpha + \int_0^T w(\alpha)g(\alpha)d\alpha \\ &= E + w = y \end{aligned}$$

w is a Gaussian random variable with mean and variance

$$E[w] = E\left[\int_0^T w(\alpha)g(\alpha)d\alpha\right] = \int_0^T E[w(\alpha)]g(\alpha)d\alpha = 0$$

$$\begin{aligned} \sigma_w^2 = E[w^2] &= E\left[\int_0^T w(\alpha)g(\alpha)d\alpha \int_0^T w(\beta)g(\beta)d\beta\right] \\ &= \int_0^T \int_0^T E\left[w(\alpha)w(\beta)\right]\ g(\alpha)g(\beta)d\alpha d\beta \end{aligned}$$

Binary Signaling

$$\begin{aligned}\sigma_w^2 &= \int_0^T\int_0^T \phi_{ww}(\alpha-\beta)g(\alpha)g(\beta)d\alpha d\beta \\ &= \frac{N_o}{2}\int_0^T\int_0^T \delta(\alpha-\beta)g(\alpha)g(\beta)d\alpha d\beta \\ &= \frac{N_o}{2}\int_0^T g^2(\alpha)d\alpha = \frac{N_oE}{2}\end{aligned}$$

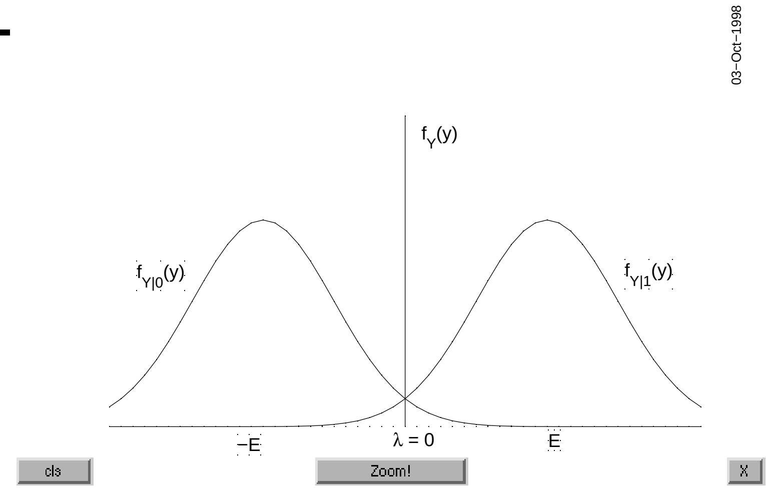
Therefore, given that '1' was sent, $y = y(T)$ has the conditional pdf

$$f_{y|`1'}(y|`1') \sim N(E, N_oE/2) = \frac{1}{\sqrt{2\pi}\sigma_w}\exp\{\frac{-(y-E)^2}{2\sigma_w^2}\}, \quad \sigma_w^2 = \frac{N_oE}{2}$$

Likewise, given that '0' was sent, $y = y(T)$ has the conditional pdf

$$f_{y|`0'}(y|`0') \sim N(-E, N_oE/2) = \frac{1}{\sqrt{2\pi}\sigma_w}\exp\{\frac{-(y+E)^2}{2\sigma_w^2}\}, \quad \sigma_w^2 = \frac{N_oE}{2}$$

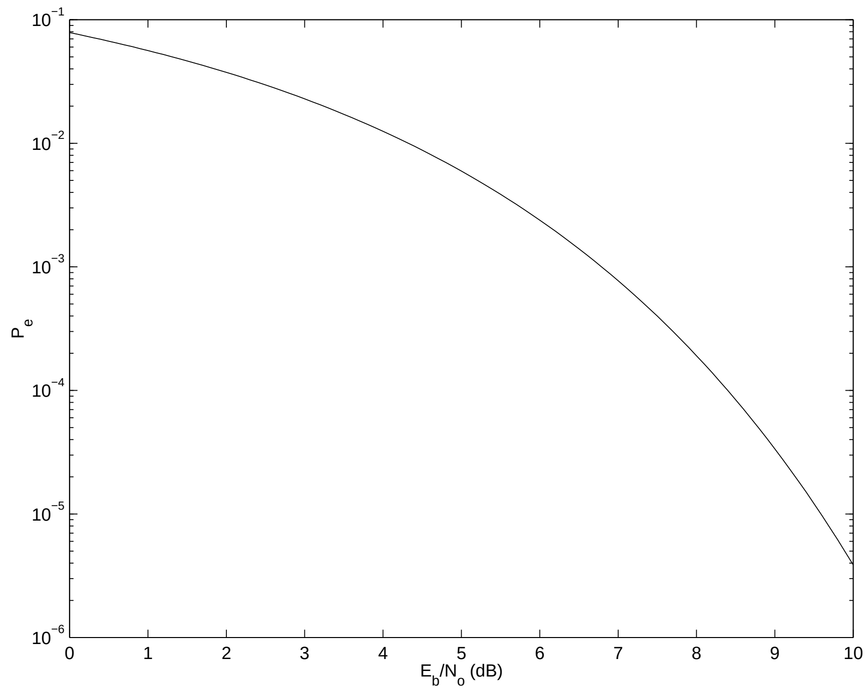
Probability of Error



$$
\begin{aligned}
P_e &= P_{e|`1'}P(`1') + P_{e|`0'}P(`0') \\
&= P(y < 0|`1')P(`1') + P(y > 0|`0')P(`0') \\
&= Q\left(\frac{E}{\sqrt{\frac{N_oE}{2}}}\right)\cdot\frac{1}{2} + Q\left(\frac{E}{\sqrt{\frac{N_oE}{2}}}\right)\cdot\frac{1}{2} = Q\left(\sqrt{\frac{2E}{N_o}}\right)
\end{aligned}
$$

If $P(`1') \neq P(`0')$, then $\lambda = 0$ <u>does not</u> yield the smallest P_e.

Probability of Error



Example: On-Off keying

$$\begin{aligned} \text{Let } `1' &\rightarrow g(t) \\ `0' &\rightarrow 0 \text{ transmit nothing} \end{aligned}$$

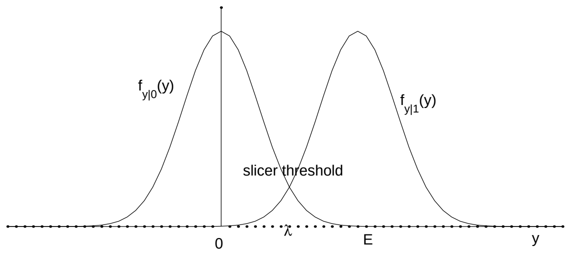
As before we use a filter matched to $g(t)$ and sample the output
If '1' is sent then

$$\begin{aligned} y &= E + w \quad w \sim N(0, N_o E/2) \\ f_{y|`1'} &\sim N(E, N_o E/2) \end{aligned}$$

If '0' is sent then

$$\begin{aligned} y &= 0 + w \quad w \sim N(0, N_o E/2) \\ f_{y|`0'} &\sim N(0, N_o E/2) \end{aligned}$$

On-Off Keying



For equally likely binary singnals, the optimum slicer threshold (minizes P_e) is where the conditional pdfs cross. In this case $\lambda = E/2$.

$$
\begin{aligned}
P_e &= P_{e|`1'}P(`1') + P_{e|`0'}P(`0') \\
&= P(y < E/2|`1')P(`1') + P(y > E/2|`0')P(`0') \\
&= Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right)\frac{1}{2} + Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right)\frac{1}{2} \\
&= Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right) = Q\left(\sqrt{\frac{\bar{E}}{N_o}}\right)
\end{aligned}
$$

$\bar{E} = E/2$ - average bit energy

On-Off Keying

$P_e = Q\left(\sqrt{\frac{\bar{E}}{N_o}}\right)$ for On-Off keying Recall $P_e = Q\left(\sqrt{\frac{2E}{N_o}}\right)$ for antipodal signals