EE4601
Communication Systems

Week 6
Orthogonal Expansions

Basic Problem

Problem:

Suppose that we have a set of M finite energy signals $S = \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where each signal has a duration T seconds.

Every T seconds one of the waveforms from the set S is selected for transmission over an AWGN channel. The transmitted waveform is

$$x(t) = \sum_n s_n(t - nT)$$

The received noise corrupted waveform is

$$r(t) = \sum_n s_n(t - nT) + n(t)$$

By observing $r(t)$ we wish to determine the *time sequence* of waveforms $\{s_n(t)\}$ that was transmitted. That is, in each T second interval, we must determine which $s_i(t) \in S$ was transmitted.

Orthogonal Expansions

Consider a real valued signal $s(t)$ with finite energy E_s,

$$E_s = \int_{-\infty}^{\infty} s^2(t)dt$$

Suppose there exists a set of *orthornormal* functions $\{f_n(t)\}, n = 1, \ldots, N$. By orthornormal we mean

$$\int_{-\infty}^{\infty} f_n(t) f_k(t) dt = \delta_{kn} \qquad \delta_{kn} = \begin{cases} 1 & , \quad k = n \\ 0 & , \quad k \neq n \end{cases}$$

We now approximate $s(t)$ as the weighted linear sum

$$\hat{s}(t) = \sum_{k=1}^{N} s_k f_k(t)$$

and wish to determine the $s_k, k = 1, \ldots, N$ to minimize the square error

$$\begin{aligned} \varepsilon &= \int_{-\infty}^{\infty} \left(s(t) - \hat{s}(t)\right)^2 dt \\ &= \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^{N} s_k f_k(t)\right)^2 dt \end{aligned}$$

Orthogonal Expansions

To minimize the mean square error, we take the partial derivative with respect to each of the s_k and set equal to zero, i.e., for the nth term we solve

$$\frac{\partial \varepsilon}{\partial s_n} = 2\int_{-\infty}^{\infty}\left(s(t) - \sum_{k=1}^{N} s_k f_k(t)\right) f_n(t)dt = 0.$$

Using the orthonormal property of the basis functions, $s_n = \int_{-\infty}^{\infty} s(t)f_n(t)dt$ and

$$\begin{aligned}
\varepsilon &= \int_{-\infty}^{\infty}\left(s(t) - \sum_{k=1}^{N} s_k f_k(t)\right)^2 dt \\
&= \int_{-\infty}^{\infty} s^2(t)dt - 2\int_{-\infty}^{\infty} s(t)\sum_{k=1}^{N} s_k f_k(t)dt + \int_{-\infty}^{\infty}\sum_{k=1}^{N} s_k f_k(t)\sum_{\ell=1}^{N} s_\ell f_\ell(t)dt \\
&= \int_{-\infty}^{\infty} s^2(t)dt - 2\sum_{k=1}^{N} s_k\int_{-\infty}^{\infty} s(t)f_k(t)dt + \sum_{k=1}^{N}\sum_{\ell=1}^{N} s_k s_\ell\int_{-\infty}^{\infty} f_k(t)f_\ell(t)dt \\
&= E_s - \sum_{k=1}^{N} s_k^2
\end{aligned}$$

For a complete set of basis functions $\varepsilon = 0$.

Gram-Schmidt Orthonormalization

Suppose that we have a set of finite energy real signals $\{s_i(t)\}, i = 1, \ldots, M\}$. We wish to obtain a *complete* set of orthonormal basis functions for the signal set. This can be done in 2 steps.

Step1: Determine if the set of waveforms is linearly independent. If they are linearly dependent, then there exists a set of coefficients $a_1, a_2 \ldots, a_M$, not all zero, such that

$$a_1 s_1(t) + a_2 s_2(t) + \cdots + a_M s_M(t) = 0.$$

Suppose, without loss of generality, that $a_M \neq 0$. If $a_M = 0$, then the signal set can be permuted so that $a_M \neq 0$. Then

$$s_M(t) = -\left(\frac{a_1}{a_M} s_1(t) + \frac{a_2}{a_M} s_2(t) + \cdots + \frac{a_{M-1}}{a_M} s_M(t)\right) \ .$$

Next consider the reduced signal set $\{s_i(t)\}_{i=1}^{M-1}$. If this set of waveforms is linearly dependent, then there exists another set of co-efficients $\{b_i\}_{i=1}^{M-1}$, not all zero, such that

$$b_1 s_1(t) + b_2 v_2(t) + \cdots + b_{M-1} s_{M-1}(t) = 0 \ .$$

Gram-Schmidt Orthonormalization

We continue until a set $\{s_i(t)\}_{i=1}^N$ of linearly independent waveforms is obtained. Note that $N \leq M$ with equality if and only if the set of waveforms $\{s_i(t)\}_{i=1}^M$ is linearly independent.

If $N < M$, then the set of linearly independent waveforms $\{s_i(t)\}_{i=1}^N$ is not unique, but any one will do.

Step 2: From the set $\{s_i(t)\}_{i=1}^N$ construct the set of N orthonormal basis functions $\{f_i(t)\}_{i=1}^N$ as follows. First, let

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

where E_1 is the energy in the waveform $s_1(t)$, given by

$$E_1 = \int_0^T s_1^2(t)dt$$

Then

$$s_1(t) = \sqrt{E_1} f_1(t) = s_{11} f_1(t)$$

where $s_{11} = \sqrt{E_1}$.

Gram-Schmidt Orthonormalization

Next, by using the waveform $s_2(t)$ we obtain

$$s_{21} = \int_0^T s_2(t) f_1(t) dt$$

along with the intermediate function

$$g_2(t) = s_2(t) - s_{21} f_1(t)$$

Note that $g_2(t)$ is orthogonal to $f_1(t)$.
The second basis function is

$$\begin{aligned} f_2(t) &= \frac{g_2(t)}{\sqrt{\int_0^T (g_2(t))^2 dt}} \\ &= \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \end{aligned}$$

Gram-Schmidt Orthonormalization

Continuing in the above fashion, we define the ith intermediate function

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} f_j(t)$$

where

$$s_{ij} = \int_0^T s_i(t) f_j(t) dt$$

The set of functions

$$f_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T (g_i(t))^2}} \qquad i = 1,\ 2,\ ,\ldots,\ N$$

is the required set of complete orthonormal basis functions.

Gram-Schmidt Orthonormalization

We can now write the signals as weighted linear combinations of the basis functions, i.e.,

$$\begin{aligned}
s_1(t) &= s_{11}f_1(t) \\
s_2(t) &= s_{21}f_1(t) + s_{22}f_2(t) \\
s_3(t) &= s_{31}f_1(t) + s_{32}f_2(t) + f_{33}f_3(t) \\
\vdots &= \vdots \\
s_N(t) &= s_{N1}f_1(t) + \cdots + s_{NN}f_N(t)
\end{aligned}$$

For the remaining signals $s_i(t), i = N+1, \ldots, M$, we have

$$s_i(t) = \sum_{k=1}^{N} s_{ik} f_k(t)$$

where

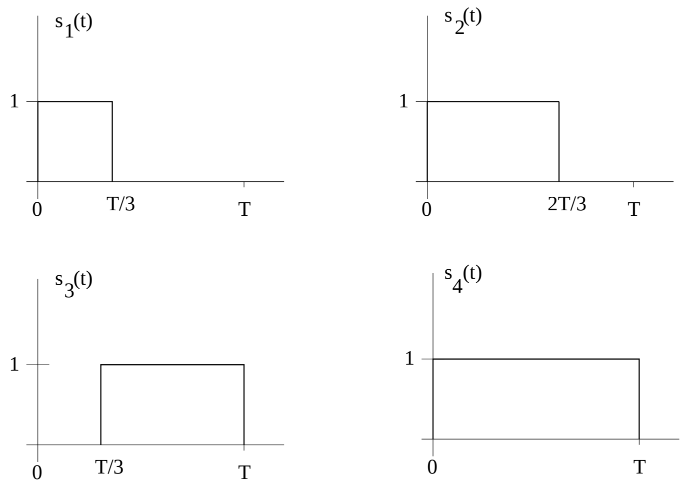
$$s_{ik} = \int_0^T s_i(t) f_k(t) dt$$

Signal Vectors

It follows that the signal set $s_i(t), i = 1, \ldots, M$ can be expressed in terms of a set of signal vertors $\mathbf{s}_i, i = 1, \ldots, M$ in an N-dimensional signal space, i.e.,

$$\begin{aligned}
s_1(t) \leftrightarrow \mathbf{s}_1 &= (s_{11}, s_{12}, \ldots, s_{1N}) \\
s_2(t) \leftrightarrow \mathbf{s}_2 &= (s_{21}, s_{22}, \ldots, s_{2N}) \\
\vdots &= \vdots \\
s_M(t) \leftrightarrow \mathbf{s}_M &= (s_{M1}, s_{M2}, \ldots, s_{MN})
\end{aligned}$$

Example



Example

Step 1: This signal set is not linearly independent because

$$s_4(t) = s_1(t) + s_3(t)$$

Therefore, we will use $s_1(t)$, $s_2(t)$, and $s_3(t)$ to obtain the complete orthonormal set of basis functions.
Step 2:
a)

$$E_1 = \int_0^T s_1^2(t)dt = T/3$$

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \sqrt{3/T} \quad , & 0 \le t \le T/3 \\ 0 \quad , & \text{else} \end{cases}$$

Example

b)

$$
\begin{aligned}
s_{21} &= \int_0^T s_2(t) f_1(t) dt \\
&= \int_0^{T/3} \sqrt{3/T} dt = \sqrt{T/3}
\end{aligned}
$$

$$E_2 = \int_0^T s_2^2(t) dt = 2T/3$$

$$
\begin{aligned}
f_2(t) &= \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \\
&= \begin{cases} \sqrt{3/T} \quad , & T/3 \le t \le 2T/3 \\ 0 \quad , & \text{else} \end{cases}
\end{aligned}
$$

Example

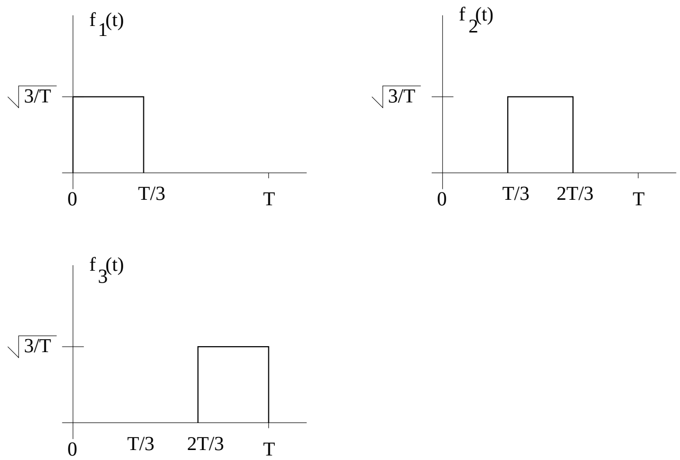
c)

$$\begin{aligned} s_{31} &= \int_0^T s_3(t) f_1(t) dt = 0 \\ s_{32} &= \int_0^T s_3(t) f_2(t) dt \\ &= \int_{T/3}^{2T/3} \sqrt{3/T} dt = \sqrt{T/3} \end{aligned}$$

$$\begin{aligned} g_3(t) &= s_3(t) - s_{31} f_1(t) - s_{32} f_2(t) \\ &= \begin{cases} 1 \; , & 2T/3 \le t \le T \\ 0 \; , & \text{else} \end{cases} \end{aligned}$$

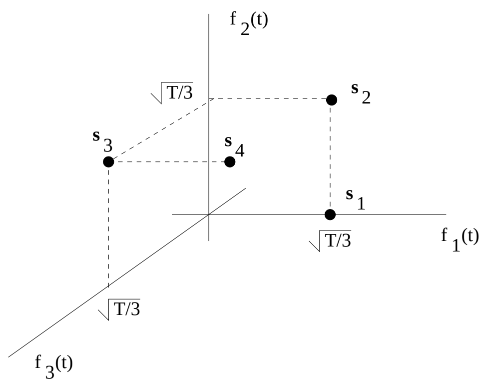
$$\begin{aligned} f_3(t) &= \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t) dt}} \\ &= \begin{cases} \sqrt{3/T} \; , & 2T/3 \le t \le T \\ 0 \; , & \text{else} \end{cases} \end{aligned}$$

Example



Example

$$
\begin{aligned}
s_1(t) &\leftrightarrow \mathbf{s}_1 = (\sqrt{T/3}, 0, 0) \\
s_2(t) &\leftrightarrow \mathbf{s}_2 = (\sqrt{T/3}, \sqrt{T/3}, 0) \\
s_3(t) &\leftrightarrow \mathbf{s}_3 = (0, \sqrt{T/3}, \sqrt{T/3}) \\
s_4(t) &\leftrightarrow \mathbf{s}_4 = (\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3})
\end{aligned}
$$



Properties of Signal Vectors

Signal Energy:

$$\begin{aligned}
E &= \int_0^T s^2(t)dt \\
&= \int_0^T \sum_{k=1}^{N} s_k f_k(t) \sum_{\ell=1}^{N} s_\ell f_\ell \; dt \\
&= \sum_{k=1}^{N} \sum_{\ell=1}^{N} s_k s_\ell \int_0^T f_k(t) f_\ell(t) dt \\
&= \sum_{k=1}^{N} s_k^2 \\
&\triangleq \|\mathbf{s}\|^2
\end{aligned}$$

The energy in $s(t)$ is just the squared length of its signal vector $\mathbf{s}$.

Properties of Signal Vectors

Signal Correlation: The correlation or "similarity" between two signals $s_j(t)$ and $s_k(t)$ is

$$\begin{aligned}
\rho_{jk} &= \frac{1}{\sqrt{E_j E_k}} \int_0^T s_j(t) s_k(t) dt \\
&= \frac{1}{\sqrt{E_j E_k}} \int_0^T \sum_{n=1}^{N} s_{jn} f_n(t) \sum_{m=1}^{N} s_{km} f_m(t)\ dt \\
&= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^{N} \sum_{m=1}^{N} s_{jn} s_{km} \int_0^T f_n(t) f_m(t) dt \\
&= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^{N} s_{jn} s_{kn} \\
&= \frac{\mathbf{s}_j \cdot \mathbf{s}_k}{\|\mathbf{s}_j\|\ \|\mathbf{s}_k\|}
\end{aligned}$$

Note that

$$\rho = \begin{cases} 0 \ , & \text{if } s_j(t) \text{ and } s_k(t) \text{ are orthogonal} \\ \pm 1 \ , & \text{if } s_j(t) = \pm s_k(t) \end{cases}$$

Properties of Signal Vectors

Euclidean Distance: The Euclidean distance between two signals $s_j(t)$ and $s_k(t)$ is

$$\begin{aligned} d_{jk} &= \left\{\int_0^T (s_j(t) - s_k(t))^2 dt\right\}^{1/2} \\ &= \left\{\int_0^T \left(\sum_{n=1}^N s_{jn} f_n(t) - \sum_{m=1}^N s_{km} f_m(t)\right)^2 dt\right\}^{1/2} \\ &= \left\{\sum_{n=1}^N (s_{jn} - s_{kn})^2\right\}^{1/2} \\ &= \left\{\|\mathbf{s}_j - \mathbf{s}_k\|^2\right\}^{1/2} \\ &= \|\mathbf{s}_j - \mathbf{s}_k\| \end{aligned}$$

Example

Consider the earlier example where

$$\begin{aligned}\mathbf{s}_1 &= (\sqrt{T/3}, 0, 0)\\ \mathbf{s}_2 &= (\sqrt{T/3}, \sqrt{T/3}, 0)\\ \mathbf{s}_3 &= (0, \sqrt{T/3}, \sqrt{T/3})\end{aligned}$$

We have $E_1 = \|\mathbf{s}_1\|^2 = T/3$, $E_2 = \|\mathbf{s}_2\|^2 = 2T/3$, and $E_3 = \|\mathbf{s}_3\|^2 = 2T/3$. The correlation between $s_2(t)$ and $s_3(t)$ is

$$\rho_{23} = \frac{\mathbf{s}_2 \cdot \mathbf{s}_3}{\|\mathbf{s}_2\|\|\mathbf{s}_3\|} = \frac{T/3}{2T/3} = 0.5$$

The Euclidean distance between $s_1(t)$ and $s_3(t)$ is

$$d_{13} = \|\mathbf{s}_1 - \mathbf{s}_3\| = \sqrt{T/3 + T/3 + T/3} = \sqrt{T}$$