EE4061
Communication Systems

Week 8

Coherent Signal Detection

Optimum Coherent Detection

Suppose that we have M signals $s_1(t), s_2(t), \ldots, s_M(t)$ that are defined over the time interval $0 \leq t \leq T$, where T is the baud period or baud interval. The *baud rate* is $R = 1/T$.

Note that $T_b = T/\log_2 M$ and $R = R_b/\log_2 M$, where $R_b = 1/T_b$ is the *bit rate*.

We select one of signals, say $s_i(t)$, for transmission over an AWGN channel.

The received signal is

$$r(t) = s_i(t) + n(t)$$

where $n(t)$ is AWGN with power spectral density $N_o/2$ watts/Hz.

Problem: By observing $r(t)$ determine which one of the M signals in the set $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ was (most likely) transmitted.

Repeat this process once every T seconds for each modulated signal that is transmitted.

Correlation Detector

Any signal set $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ can be expressed in terms of a set of orthonormal basis functions $\{f_1(t), f_2(t), \ldots, f_N(t)\}$ where N is the dimension of the signal space. Recall the Gram-Schmidt orthonormalization procedure.

However, the basis functions *do not* span the noise space, i.e., the noise waveform $n(t)$ cannot be represented exactly in terms of the N basis functions. We have

$$r(t) = \sum_{k=1}^{N} r_i f_i(t) + \hat{n}(t)$$

where

$$\begin{aligned} r_k &= \int_0^T r(t) f_k(t) dt \\ &= \int_0^T s_m(t) f_k(t) dt + \int_0^T n(t) f_k(t) dt \\ &= s_{mk} + n_k \end{aligned}$$

Hence, the projection of the received signal $r(t)$ onto the *signal space* yields the *received* vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$.

Remainder Process

Note that

$$r(t) = \sum_{k=1}^{N} r_k f_k(t) + \hat{n}(t)$$

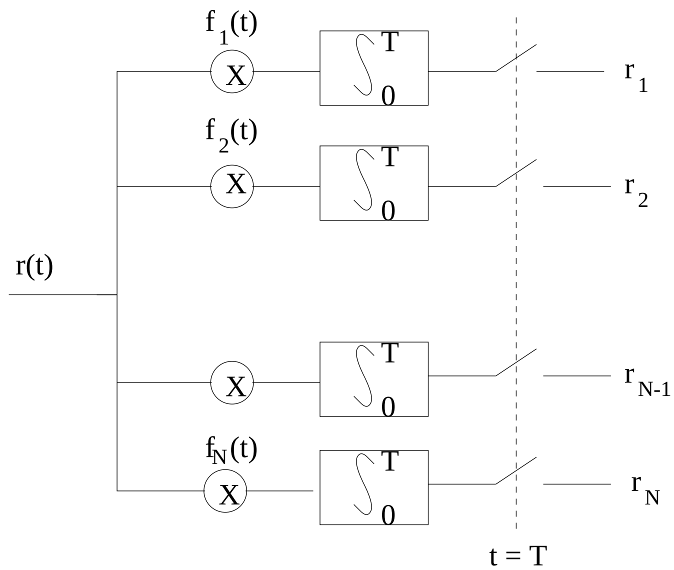
where $\hat{n}(t)$ is the **remainder process**

$$\hat{n}(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)$$

We will see later that the remainder process $\hat{n}(t)$ is irrelevant when deciding as to which $s_i(t)$ was sent.

The vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ is said to provide *sufficient statistics*, meaning that nothing else is required other than the vector $\mathbf{r}$ for the receiver to decide which $s_i(t)$ was sent.

Correlation Detector



Noise Statistics

The noise components n_k have mean

$$E[n_k] = \int_0^T E[n(t)] f_k(t) dt = 0$$

and covariance

$$\begin{aligned} E[n_j n_k] &= \int_0^T \int_0^T E[n(t)n(s)] f_j(t) f_k(s) dt ds \\ &= \frac{N_o}{2} \int_0^T \int_0^T \delta(t-s) f_j(t) f_k(s) dt ds \\ &= \frac{N_o}{2} \int_0^T f_j(t) f_k(t) dt \\ &= \frac{N_o}{2} \delta_{jk} \end{aligned}$$

Therefore, the r_k are independent Gaussian random variables with mean s_{mk} and variance $N_o/2$.

Joint Conditional Density

Since elements of the vector $\mathbf{r}$ are independent random variables the joint conditional density function of the vector $\mathbf{r}$ has the product form

$$\begin{aligned}
p(\mathbf{r}|\mathbf{s_m}) &= \prod_{k=1}^{N} p(r_k|s_{mk}) \\
&= \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_o)}} \exp\left\{-\frac{1}{N_o}(r_k - s_{mk})^2\right\} \\
&= \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\sum_{k=1}^{N}(r_k - s_{mk})^2\right\} \\
&= \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\|\mathbf{r} - \mathbf{s}_m\|^2\right\}
\end{aligned}$$

which is a multivariate Gaussian distribution.

Note that we have used the notation

$$\|\mathbf{r} - \mathbf{s}_m\|^2 = \sum_{k=1}^{N}(r_k - s_{mk})^2$$

Irrelevance

We have

$$\begin{aligned}
\mathrm{E}[\hat{n}(t) r_k] &= \mathrm{E}[\hat{n}(t)] s_{mk} + \mathrm{E}[\hat{n}(t) n_k] \\
&= \mathrm{E}[\hat{n}(t) n_k] \\
&= \mathrm{E}\left[\left(n(t) - \sum_{j=1}^{N} n_j f_j(t)\right) n_k\right] \\
&= \int_0^T \mathrm{E}[n(t) n(\tau)] f_k(\tau) d\tau - \sum_{j=1}^{N} \mathrm{E}[n_k n_j] f_j(t) \\
&= \frac{1}{2} N_o f_k(t) - \frac{1}{2} N_o f_k(t) = 0
\end{aligned}$$

Hence, the vector $\mathbf{r}$ is uncorrelated with $\hat{n}(t)$ and, therefore, $\hat{n}(t)$ is irrelevant since it does not contain any information about $\mathbf{r}$.

This is Wozencraft's irrelevance theorem which is certainly not irrelevant!

Matched Filter Receiver

Suppose that we filter the received signal $r(t)$ with a bank of matched filters having the impulse responses

$$h_k(t) = f_k(T-t) \quad , \quad 0 \leq t \leq T$$

and sample the filter outputs at time $t = T$.

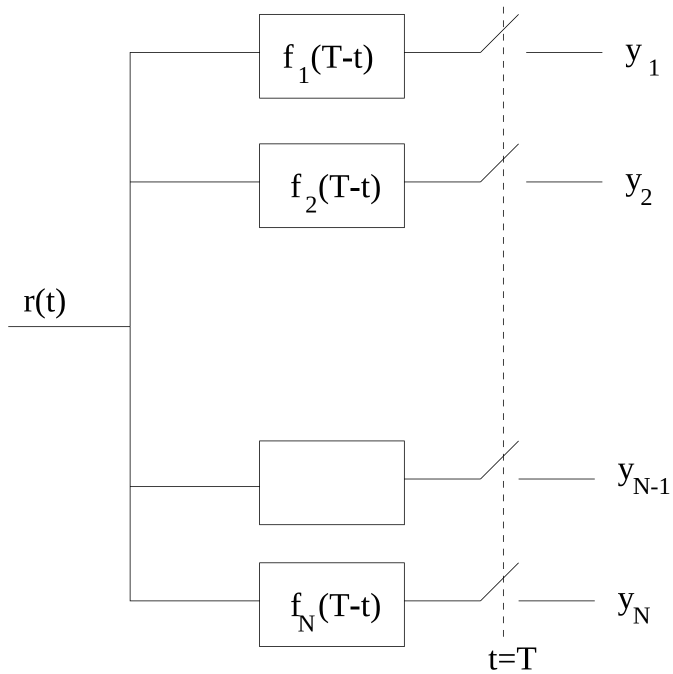
The filter outputs are

$$\begin{aligned} y_k(t) &= \int_0^t r(\tau) h_k(t-\tau) d\tau \\ &= \int_0^t r(\tau) f_k(T-t+\tau) d\tau \end{aligned}$$

$$y_k \equiv y_k(T) = \int_0^T r(\tau) f_k(\tau) d\tau$$

Note that $y_k = r_k$, i.e., the matched filter outputs are identical to the correlator outputs.

Matched Filter Receiver



Minimum Distance Decisions

With minimum distance decisions, the receiver first calculates the vector $\mathbf{r}$. The receiver then decides in favour of the signal point $\mathbf{s}_i$ that is closest in *Euclidean distance* or *squared Euclidean distance* to the received vector $\mathbf{r}$.

The minimum distance decision rule is

$$\hat{\mathbf{s}} = \underset{\mathbf{s}_i}{\arg\min} \|\mathbf{r} - \mathbf{s}_i\|^2$$

Since, the vector $\mathbf{r}$ has the joint conditional density function

$$p(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\|\mathbf{r} - \mathbf{s}_m\|^2\right\}$$

the choice of $\mathbf{s}_i$ that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$ also maximizes the likelihood $p(\mathbf{r}|\mathbf{s}_m)$.

Hence *minimum distance* decisions are *maximum likelihood* decisions.