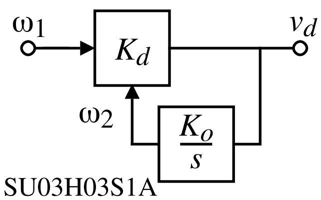
Homework Assignment No. 3 - Solutions

Problem 1 - (10 points)

Assume an LPLL has $F(s)=1$ and the PLL parameters are $K_d = 0.8$V/radians, $K_o = 100$ MHz/V, and the oscillation frequency, $f_{osc} = 500$MHz. Sketch the control voltage at the output of the phase detector if the input frequency jumps from 500MHz to 650MHz.

Solution

Find the transfer function from the input frequency, f_{in}, to the output of the phase detector, v_d.



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$$V_d = K_d(\theta_1-\theta_2) = K_d\theta_1 - \frac{K_dK_o}{s}V_d$$

$$V_d\left(1+\frac{K_dK_o}{s}\right) = K_d\theta_1 = K_d\left(\frac{\omega_1}{s}\right)$$

$$\therefore \quad \frac{V_d}{\omega_1} = \frac{K_d}{s+K_dK_o} \rightarrow \qquad V_d(s) = \frac{K_d}{s+K_dK_o}\omega_1(s) = \frac{K_d}{s+K_dK_o}\frac{\Delta\omega_1}{s} = \frac{k_1}{s} + \frac{k_2}{s+K_dK_o}$$

By partial fraction expansion we can show that $k_1 = -k_2 = \frac{K_d\Delta\omega_1}{K_dK_o} = \frac{K_d\Delta\omega_1}{K_v} = 1.5\text{V}$

Note the units of $\frac{K_d\Delta\omega_1}{K_v}$ are $\frac{\text{(V/rad)(rad/sec)}}{\text{1/sec}} = \text{V}$

and $K_v = (2\pi\cdot 100\text{MHz/V})(0.8\text{V/rad.}) = 502.65\text{x}10^6$ (1/sec.)

$$\therefore \quad v_d(t) = \frac{K_d\Delta\omega_1}{K_v}(1-e^{-K_vt}) = 1.5(1-e^{-502.65\text{x}10^6t})$$

A plot of $v_d(t)$ is shown below.

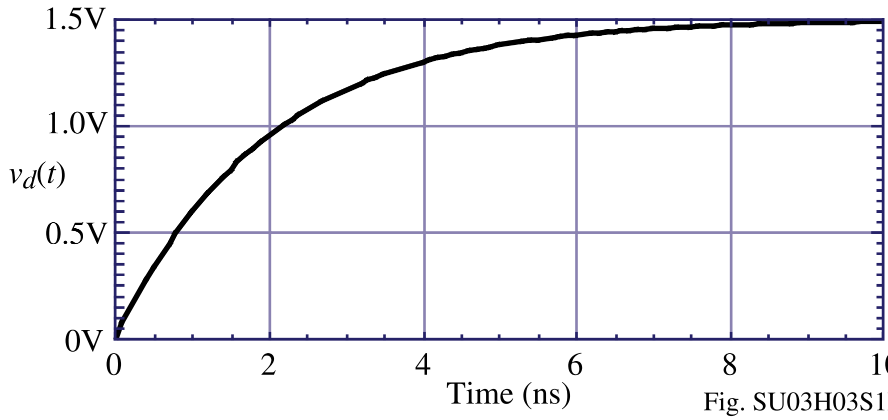
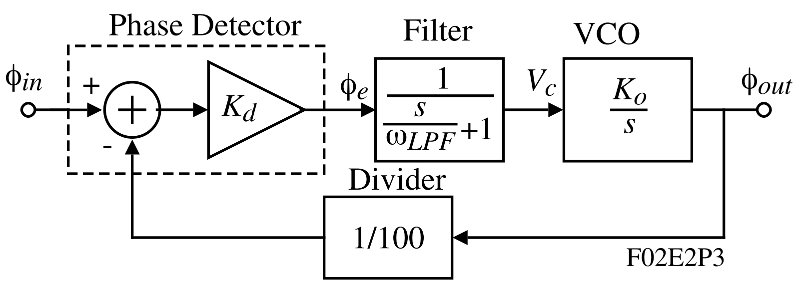


Fig. SU03H03S1B

Problem 2 – (10 points)

A Type I PLL incorporates a VCO with K_o = 100MHz/V, a phase detector with K_d = 1V/rad, and a first-order, lowpass filter with $\omega_{LPF} = 2\pi \times 10^6$ radians/s shown below. A divider of 100 has been placed in the feedback path to implement a frequency synthesizer. (a.) Find the value of the natural damping frequency, ω_n, and the damping factor, ζ, for the transfer function $\phi_{out}(s)/\phi_{in}(s)$, for this PLL. (b.) If a step input of $\Delta\phi_{in}$ is applied at $t = 0$, what is the steady-state phase error at the output of the phase detector, ϕ_e? The steady-state error is evaluated by multiplying the desired phase by s and letting $s \to 0$.



Solution

(a.) $$\phi_{out} = \frac{K_o}{s}\left(\frac{1}{\frac{s}{\omega_{LPF}}+1}\right)K_d\left(\phi_{in} - \frac{\phi_{out}}{N}\right) \rightarrow \phi_{out}\left[1 + \frac{K_o}{sN}\left(\frac{K_d}{1+\frac{s}{\omega_{LPF}}}\right)\right] = \frac{K_o}{s}\left(\frac{K_d}{\frac{s}{\omega_{LPF}}+1}\right)\phi_{in}$$

$$\therefore \quad \frac{\phi_{out}(s)}{\phi_{in}(s)} = \frac{K_oK_d}{s\left(1+\frac{s}{\omega_{LPF}}\right)+\frac{K_oK_d}{N}} = \frac{K_oK_d\omega_{LPF}}{s^2+\omega_{LPF}\,s+\frac{K_oK_d\omega_{LPF}}{N}} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Thus, $$\omega_n^2 = \frac{K_oK_d\omega_{LPF}}{N} = \frac{2\pi x10^6\cdot 2\pi x10^8}{100} = 4\pi^2 x10^{12} \rightarrow \omega_n = 2\pi x10^6$$

$$\zeta = \frac{\omega_{LPF}}{2\omega_n} = \frac{\omega_{LPF}}{2\sqrt{\frac{K_oK_d\omega_{LPF}}{N}}} = \frac{1}{2}\sqrt{\frac{N\omega_{LPF}}{K_oK_d}} = \frac{1}{2}\sqrt{\frac{100\cdot 2\pi x10^6}{1\cdot 2\pi x10^8}} = 0.5$$

$\therefore$ $\boxed{\omega_n = 2\pi x10^6 \text{ and } \zeta = 0.5}$

(b.) First we must solve for $\phi_e(s)$ which is found as

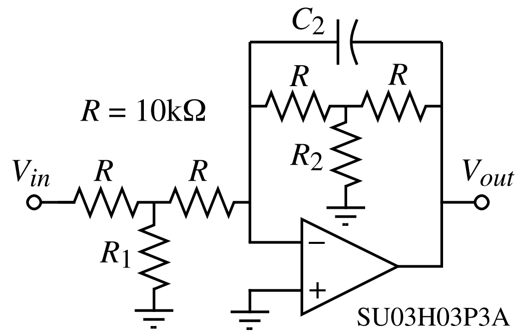
$$\phi_e(s) = \frac{s\left(1+\frac{s}{\omega_{LPF}}\right)}{K_o}\phi_{out}(s) = \frac{s\left(1+\frac{s}{\omega_{LPF}}\right)}{K_o}\frac{K_oK_d\omega_{LPF}}{s^2+\omega_{LPF}\,s+\frac{K_oK_d\omega_{LPF}}{N}}\phi_{in}(s)$$

If $\phi_{in}(s) = \frac{\Delta\phi_{in}}{s}$, then we can write $s\phi_e(s) = \dfrac{K_d(s^2+\omega_{LPF}\,s)\Delta\phi_{in}}{s^2+\omega_{LPF}\,s+\frac{K_oK_d\omega_{LPF}}{N}}$

Therefore, we see that the steady-state error is $\boxed{\phi(t=\infty) = 0.}$

Problem 3 – (10 points)

Modify the active filter shown of Problem 4 of Homework 2 to design the lag-lead loop filter shown below. The capacitors can be no larger than 10pF. Give the values of R_1, R_2, C_1 and C_2.



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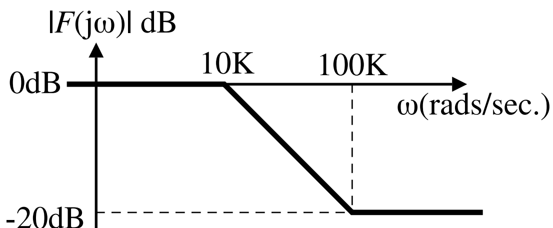

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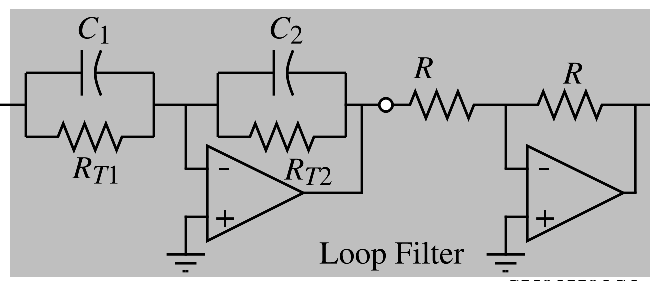
Solution

The transfer function corresponding to the above Bode plot is,

$$F(s) = \frac{\frac{s}{10^5}+1}{\frac{1}{10^4}+1}$$

The modification of the filter is shown where from Prob. 4 of Homework 2,



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$$R_{Ti} = \frac{2RR_i + R^2}{R_i}$$

The transfer function of this filter is found as,

$$F(s) = \frac{V_c(s)}{V_d(s)} = \left(\frac{R_{T2}}{R_{T1}}\right)\frac{sR_{T1}C_1+1}{sR_{T2}C_2+1} \Rightarrow \quad R_{T2} = R_{T1} = R_T\ ,\ R_TC_1 = 10^{-5} \text{ and } R_TC_2 = 10^{-4}$$

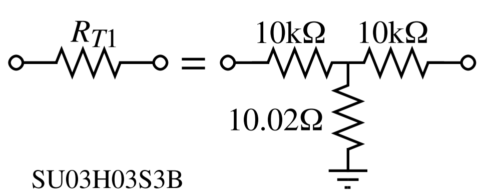
We see if $R_{T2} = R_{T1}$, then $C_2 = 10C_1$. Choosing C_2 = 10pF gives C_1 = 1pF. This gives

$$R_T = \frac{10^{-4}}{C_2} = \frac{10^{-4}}{10^{-11}} = 10^7$$

$$R_T = \frac{2RR_i + R^2}{R_i} = 2R + \frac{R^2}{R_1} = 20\text{x}10^3 + \frac{100\text{x}10^6}{R_1} = 10^7 \Rightarrow R_1 = \frac{100\text{x}10^6}{10^7\text{-}20\text{x}10^3} = 10.02\Omega$$

Therefore, $R_1 = R_2$ = 10.02Ω, C_1 = 1pF and C_2 = 10pF

The realization is completed by replacing each of the R_T resistors with the following equivalent:



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Problem 4 – (10 points)

Using the filter of Problem 3, find the value of ω_n and ζ of the PLL if K_d = 1V/radians, K_o = 2Mradians/V·sec. What is the steady state phase error in degrees if a frequency ramp of 10^9 radians/sec.^{2} is applied to the PLL?

Solution

Using the definition give in the notes for the time constants of the passive lag-lead filter we get,

$$F(s) = \frac{\frac{s}{10^5}+1}{\frac{1}{10^4}+1} = \frac{s\tau_2 + 1}{s(\tau_1+\tau_2) + 1} \quad \Rightarrow \quad \tau_2 = 10^{-5} \text{ sec. and } \tau_1 = 9\text{x}10^{-5} \text{ sec.}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K_oK_d}{\tau_1+\tau_2}} = \sqrt{\frac{2\text{x}10^6}{10^{-4}}} = \sqrt{2}\text{ x}10^5 = \underline{\underline{141.4\text{x}10^3 \text{ radians/sec.}}}$$

$$\zeta = \frac{\omega_n}{2}\left(\tau_2 + \frac{1}{K_oK_d}\right) = \frac{\sqrt{2}\text{x}10^5}{2}\left(10^{-5} + \frac{1}{2\text{x}10^6}\right) = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{20}\right) = \underline{\underline{0.742}}$$

Assuming the PLL has a high loop gain, then the steady-state phase error can be found as

$$\theta_e(\infty) = \frac{\Delta\dot{\omega}}{\omega_n^2} = \frac{10^9}{2\text{x}10^{10}} = \frac{1}{20} \text{ radians } = \underline{\underline{2.86°}}$$

Problem 5 – (10 points)

Solve for the crossover frequency of the PLL of Problems 3 and 4 and find the phase margin. Use SPICE to find the open-loop frequency response of the PLL and from your plot determine the crossover frequency and phase margin and compare with your calculated values.

Solution

The crossover frequency can be found as,

$$\omega_c = \omega_n\sqrt{2\xi^2 + \sqrt{4\xi^4+1}} = \sqrt{2}\ \text{x}10^5\sqrt{2\cdot 0.742^2 + \sqrt{4\cdot 0.742^4+1}}$$

$$= \sqrt{2}\ \text{x}10^5(1.6089) = 2.275\text{x}10^5 \text{ radians/sec.} = 36.208\text{kHz}$$

The open loop transfer function is given as

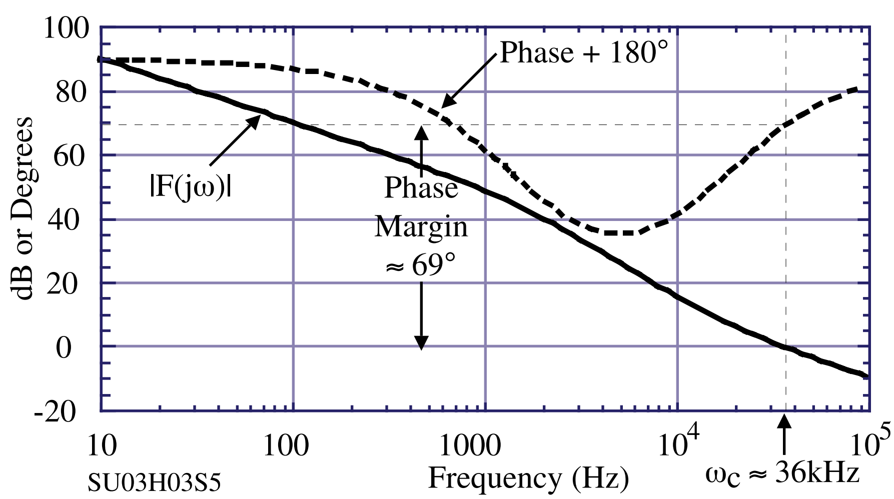
$$\text{LG}(s) = \frac{K_v}{s}\left(\frac{1+s\tau_1}{1+s\tau_2}\right) = \frac{\sqrt{2}\text{x}10^5}{s}\left(\frac{1+s10^{-5}}{1+s10^{-4}}\right)$$

The phase margin can be written as,

$$\text{PM} = 180° - 90° + \tan^{-1}\left(\frac{\omega_c}{10^5}\right) - \tan^{-1}\left(\frac{\omega_c}{10^4}\right) = 90° + 66.27° - 87.48° = \underline{\underline{68.79°}}$$

SPICE Results:

```
Problem H3P5-Open Loop Response of an LPLL with Lead-Lag Filter
VS 1 0 AC 1.0
R1 1 0 10K
* Loop bandwidth = Kv =2xE+6    Tau1=1E-4  Tau2=1E-5
ELPLL 2 0 LAPLACE {V(1)}=
+{(2E+6/(S+0.001))*((1+1E-5*S)/(1+1E-4*S))}
* Note: The 0.001 added to "S" in the denominator is to prevent
* blowup of the problem at low frequencies.
R2 2 0 10K
*Steady state AC analysis
.AC DEC 20 10 100K
.PRINT AC VDB(2) VP(2)
.PROBE
.END
```



The simulation results agree well with the calculated results.